CLASSIFYING EXPANSIONS OF THE REAL FIELD BY COMPLEX SUBGROUPS

BY

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DISSERTATION

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Abstract

In this thesis, we study expansions of the real field by multiplicative subgroups of the complex numbers. We first consider expansions by a subgroup generated by an element of the unit circle and a positive real number. We then consider expansions by a subgroup generated by a complex number and a positive real number. In both of these cases, we investigate the sets definable in these structures and their open cores.

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Conventions and notations

- By $\mathbb{N}$, we mean the set $\{0, 1, 2, \ldots\}$.
- Throughout, m and n will range over elements of $\mathbb{Z}$.
- $\mathbb{R}^{>0}$ denotes the set of positive elements of the set $\mathbb{R}$ of real numbers.
- Let K be a real closed field and let i be an element in an extension of K such that $i^2 = -1$. We will identify $K(i)$ with K^2 in the same way that we identify $\mathbb{C}$ with $\mathbb{R}^2$. That is, we identify the element $a + bi \in K(i)$ with $(a, b) \in K^2$. For an element $z \in K(i)$ with $z = a + bi$, we let $\mathrm{Re}(z) = a$, $\mathrm{Im}(z) = b$.

Logic conventions

- For a set X, $|X|$ will denote the cardinality of X.
- $\mathcal{L}_{or}$ will denote the language of ordered rings. That is, $\mathcal{L}_{or} = \{<, +, \cdot, 0, 1\}$.
- RCF will denote the $\mathcal{L}_{or}$-theory of real closed fields.
- Throughout, "definable" will mean "definable with parameters". A set that is definable without parameters is called 0-definable.
- If $\mathcal{M}$ is a structure in a language $\mathcal{L}$, then by $\mathrm{Th}(\mathcal{M})$ we mean the $\mathcal{L}$-theory of $\mathcal{M}$.
- Let $\mathcal{M}$ be an $\mathcal{L}$-structure and let S be a nonempty subset of M. We will write $\mathrm{dcl}^{\mathcal{M}}(S)$ for the definable closure of S in $\mathcal{M}$. That is,

$$\mathrm{dcl}^{\mathcal{M}}(S) = \{f(s_1, \ldots, s_n) : s_1, \ldots, s_n \in S,\ f : M^n \to M \text{ is 0-definable in } \mathcal{M}\}.$$

If $\mathcal{L}'$ is a sublanguage of $\mathcal{L}$, then we define

$$\mathrm{dcl}^{\mathcal{M}}_{\mathcal{L}'}(S) := \{f(s_1, \ldots, s_n) : s_1, \ldots, s_n \in S,\ f : M^n \to M \text{ is 0-definable in } \mathcal{M}|_{\mathcal{L}'}\}.$$

- If $\overline{x}$ is a tuple of variables, then we will write $|\overline{x}|$ for the length of the tuple $\overline{x}$. That is, if $\overline{x} = (x_1, \ldots, x_n)$, then $|\overline{x}| = n$.

- Let K be a real closed field. For $k \in K$, $|k| := \max\{k, -k\}$. In a real closed field, n denotes $1+1+\ldots+1$, n times, and $1/n$ denotes its multiplicative inverse.

- Let $\mathcal{L}$ be a language and let $\mathcal{M}$ be an $\mathcal{L}$-structure. Let $C \subseteq M$. We will write $\mathcal{L}(C)$ to denote the language consisting of $\mathcal{L}$ together with a constant symbol for each element of C.

  - Let $\mathcal{L}$ be a language expanding $\mathcal{L}_{or}$ and let $\mathcal{R} = (R, \ldots)$ be an $\mathcal{L}$-structure such that R is a real closed field. For $S \subseteq R(i)$, we will write $\mathcal{L}(S)$ for $\mathcal{L}(\mathrm{Re}(S), \mathrm{Im}(S))$.

Algebra conventions

- Throughout, all groups will be multiplicative unless stated otherwise. Thus, groups will be written using multiplicative notation unless stated otherwise.

- For a field K we let $K^\times = K \setminus \{0\}$ be its multiplicative group.

- Let G be an abelian group. For $n \geq 1$, let $G^{[n]} = \{g^n : g \in G\}$. Define $[n]G$ as

$$[n]G = \begin{cases} \left|G/G^{[n]}\right| & \text{if } \left|G/G^{[n]}\right| \text{ finite} \\ \infty & \text{otherwise} \end{cases}.$$

  That is, if there are finitely many cosets of $G^{[n]}$ in G, then $[n]G$ gives the number of cosets.

- Let E, F be fields with $E \subseteq F$, and let S be a subset of F. We will write $E(S)$ for the subfield of F obtained by adjoining the set S to E. That is, $E(S)$ is the smallest subfield of F extending E and containing S.

- Let K be a field. We let $\mathbb{S}^1(K) = \{(a, b) \in K^2 : a^2 + b^2 = 1\}$. If $K = \mathbb{R}$, then we will write $\mathbb{S}^1$ instead of $\mathbb{S}^1(\mathbb{R})$.

Chapter 1

Introduction

1.1 Background

Let $\overline{\mathbb{R}} := (\mathbb{R}, <, +, \cdot, 0, 1)$ be the field of real numbers. Our goal is to classify expansions of $\overline{\mathbb{R}}$ of the form $(\overline{\mathbb{R}}, \Gamma)$, where Γ is an infinite finitely generated subgroup of $\mathbb{C}^{\times}$, according to their definable sets.

Van den Dries initiated the study of expansions of $\overline{\mathbb{R}}$ by finitely generated subgroups of $\mathbb{C}$ by considering the structure $(\overline{\mathbb{R}}, 2^{\mathbb{Z}})$ in [10]. The main results of this paper are the following two theorems.

Theorem 1.1 (Theorem I, [10])**.** *Let A be a unary predicate symbol. Let Σ be a set of axioms expressing, for $\mathcal{L}_{or}$-structures (R, A), that R is a real closed field and A a multiplicative group of positive elements of R such that*

1. $2 \in A$, $\forall x(1 < x < 2 \to x \notin A)$.
2. $\forall x(x > 0 \to \exists y \in A(y \leq x < 2y))$.

Then Σ axiomatizes $Th(\overline{\mathbb{R}}, 2^{\mathbb{Z}})$.

For $n \geq 1$, let P_n be a unary predicate. Let λ be a unary function symbol, and let $\mathcal{L}^*_{or}$ be the language $\mathcal{L}_{or} \cup \{P_n : n \geq 1\} \cup \{\lambda\}$. Let Σ^* be the $\mathcal{L}^*_{or}$-theory consisting of Σ together with the defining axioms given by the universal closures of the following formulas:

$$P_n(x) \leftrightarrow \exists y(A(y) \wedge y^n = x)(n = 1, 2, \ldots),$$

$$x \leq 0 \to \lambda(x) = 0,$$

$$x > 0 \to A(\lambda(x)) \wedge \lambda(x) \leq x < 2\lambda(x).$$

Theorem 1.2 (Theorem II, [10])**.** *The $\mathcal{L}^*_{or}$-theory Σ^* admits quantifier elimination.*

From Theorem 1.2 and the definition of Σ^*, we obtain the following corollary.

Corollary 1.3. *Every subset of $\mathbb{R}^m$ definable in $(\overline{\mathbb{R}}, 2^{\mathbb{Z}})$ is a Boolean combination of sets defined by formulas of the form*

$$\exists y_1 \ldots \exists y_n \left(\bigwedge_{i=1}^{n} (y_i \in 2^{\mathbb{Z}}) \wedge \phi(x_1, \ldots, x_m, y_1, \ldots, y_n) \right)$$

where $\phi(x, y)$ is a quantifier free $\mathcal{L}_{or}(\mathbb{R})$-formula.

Van den Dries and Günaydın then studied expansions of $\overline{\mathbb{R}}$ by dense subgroups of $\mathbb{R}^{>0}$ with the Mann property. The Mann property is defined for subgroups G of $\mathbb{R}^{\times}$ as follows. For nonzero $a_1, \ldots, a_n \in \mathbb{Q}$ ($n \geq 1$), a nondegenerate solution in G to the equation

$$a_1 x_1 + \ldots + a_n x_n = 1 \tag{1.1}$$

is a tuple $(g_1, \ldots, g_n) \in G^n$ such that $a_1 g_1 + \ldots + a_n g_n = 1$ and $\sum_{i \in I} a_i g_i \neq 0$ for each nonempty subset I of $\{1, \ldots, n\}$. We say that G has the Mann property if every equation which has the form of Eq. (1.1) has only finitely many nondegenerate solutions in G.

An abelian group G is said to have finite rank if there is a finitely generated subgroup $H \leq G$ such that G/H is torsion. It can be shown that every finite rank multiplicative subgroup of a field of characteristic 0 has the Mann property. This follows directly from Theorem 1.1 of Evertse, Schlickewei, and Schmidt [16]. Thus, the following theorems go through when we consider a subgroup of $\mathbb{R}^{\times}$ or of $\mathbb{C}^{\times}$ with finite rank.

For a subgroup G of $\mathbb{R}^{\times}$, let $(\overline{\mathbb{R}}, G, (g)_{g \in G})$ denote the structure $(\overline{\mathbb{R}}, G)$ together with constants for each element of G. In [13], van den Dries and Günaydın found an axiomatization for structures of the form $(\overline{\mathbb{R}}, \Gamma, (\gamma)_{\gamma \in \Gamma})$, where Γ is a dense subgroup of $\mathbb{R}^{>0}$ with the Mann property. From this axiomatization, they proved the following theorem about the definable sets of $(\overline{\mathbb{R}}, \Gamma)$.

Theorem 1.4 (Theorem 7.5, [13])**.** *Let Γ be a dense subgroup of $\mathbb{R}^{>0}$ with the Mann property. Suppose that $[p]\Gamma$ is finite for each prime number p. Then every subset of $\mathbb{R}^m$ definable in $(\overline{\mathbb{R}}, \Gamma)$ is a Boolean combination of subsets of $\mathbb{R}^m$ defined in $(\overline{\mathbb{R}}, \Gamma)$ by formulas*

$$\exists y_1 \ldots \exists y_n \left(\bigwedge_{i=1}^{n} (y_i \in \Gamma) \wedge \phi(x_1, \ldots, x_m, y_1, \ldots, y_n) \right)$$

where $\phi(x, y)$ is a quantifier free $\mathcal{L}_{or}(\mathbb{R})$-formula.

We say that a group $G \subseteq \mathbb{R}^{>0}$ is discrete if every point in G is isolated. In Section 2.3.2, we prove that a group $G \subseteq \mathbb{R}^{>0}$ is discrete if and only if $G = a^{\mathbb{Z}}$ for some $a \in \mathbb{R}^{>0}$. Thus, discrete subgroups of $\mathbb{R}^{>0}$ have the Mann property. Günaydın continued studying expansions of $\overline{\mathbb{R}}$ by finitely generated subgroups in [17].

In this paper, Günaydın gave an axiomatization for structures of the form $(\overline{\mathbb{R}}, A, B, (a)_{a\in A})$, where A is a discrete subgroup of $\mathbb{R}^{>0}$ and B is a dense subgroup of $\mathbb{R}^{>0}$ such that $A \subseteq B$, B has the Mann property, and $[p]A$ and $[p]B$ are finite for each prime number p.

Günaydın also gave an axiomatization for structures of the form $(\overline{\mathbb{R}}, \Gamma, (\gamma)_{\gamma\in\Gamma})$, where Γ is an infinite subgroup of the unit circle $\mathbb{S}^1$ with the Mann property. Structures of this form were also studied independently by Belegradek and Zilber [2]. The main theorem of [2] is as follows.

Theorem 1.5 (Theorem 1.3, [2])**.** *Let Γ be an infinite subgroup of $\mathbb{S}^1$ with the Mann property. Every subset of $\mathbb{R}^m$ definable in $(\overline{\mathbb{R}}, \Gamma)$ is a Boolean combination of subsets of $\mathbb{R}^m$ defined in $(\overline{\mathbb{R}}, \Gamma)$ by formulas of the form*

$$\exists x_1 \exists y_1 \ldots \exists x_n \exists y_n \left(\bigwedge_{i=1}^{n} (x_i, y_i) \in \Gamma \wedge \phi(x_1, y_1, \ldots, x_n, y_n, v_1, \ldots, v_m) \right)$$

where $\phi(x, y, v)$ is a quantifier free $\mathcal{L}_{or}(\mathbb{R})$-formula.

Hieronymi then considered expansions of $\overline{\mathbb{R}}$ by cyclic subgroups of $\mathbb{C}$ in [19] and proved the following classification theorem for such expansions.

Theorem 1.6 (Theorem 1.6, [19])**.** *Let S be an infinite cyclic subgroup of $(\mathbb{C}^\times, \cdot)$. Then exactly one of the following holds:*

1. *$(\overline{\mathbb{R}}, S)$ defines $\mathbb{Z}$,*
2. *in any model $\mathcal{M}$ of $Th(\overline{\mathbb{R}}, S)$, every definable subset of M is the union of an open set and finitely many discrete sets,*
3. *for each $n \geq 1$, every open subset of $\mathbb{R}^n$ definable in $(\overline{\mathbb{R}}, S)$ is definable in $\overline{\mathbb{R}}$.*

It can be shown (Exercise 37.6, [20]) that if $(\overline{\mathbb{R}}, S)$ defines $\mathbb{Z}$, then $(\overline{\mathbb{R}}, S)$ defines every projective subset of $\mathbb{R}$. In particular, $(\overline{\mathbb{R}}, S)$ defines every open subset of $\mathbb{R}$. On the other hand, suppose every open definable subset of $\mathbb{R}$ in $(\overline{\mathbb{R}}, S)$ is definable in $\overline{\mathbb{R}}$. If U is an open subset of $\mathbb{R}$ definable in $(\overline{\mathbb{R}}, S)$, then U has only finitely many connected components. Therefore, (1) and (3) in the previous theorem represent two extremes for the open sets that can be defined in an expansion of $\overline{\mathbb{R}}$.

Using techniques from [10], [13], and [2], we study the definable sets in expansions of $\overline{\mathbb{R}}$ by two types of subgroups. In Chapter 5, we consider structures of the form $(\overline{\mathbb{R}}, a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}})$, where $a, \varphi \in \mathbb{R}$ and $a > 1$, $\varphi \notin \pi\mathbb{Q}$. In Chapter 6, we consider structures of the form $(\overline{\mathbb{R}}, (ae^{i\varphi})^{\mathbb{Z}} b^{\mathbb{Z}})$, where $a, b, \varphi \in \mathbb{R}$ and $a, b > 1$.

Theorem 1.6 classifies expansions of the form $(\overline{\mathbb{R}}, S)$, where S is an infinite cyclic subgroup of $\mathbb{C}^\times$, by their definable sets. This raises the question of whether this classification holds for expansions of $\overline{\mathbb{R}}$ by arbitrary finitely generated subgroups of $\mathbb{C}^\times$. Let a and φ be real numbers such that $a > 1$ and $\varphi \notin \pi\mathbb{Q}$. One of the main results of this thesis, which we prove below in Section 1.3, is that structures of the form $(\overline{\mathbb{R}}, a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}})$ do not satisfy any of (1)-(3) in the above theorem. Therefore, a new classification is needed for expansions of $\overline{\mathbb{R}}$ by finitely generated subgroups of $\mathbb{C}^\times$.

1.2 Summary of known results about expansions of $\overline{\mathbb{R}}$ by subgroups of $\mathbb{C}^\times$

In the following subsections, we summarize known results about expansions of $\overline{\mathbb{R}}$ by subgroups of $\mathbb{C}^\times$. These include some results proved by the author. Before stating the results, we will need a few definitions.

Definition 1.7 (Interdefinability)**.** Let $\mathcal{L}$ be a language and let $\mathcal{A}, \mathcal{B}$ be $\mathcal{L}$-structures with the same domain A. We say that $\mathcal{A}$ and $\mathcal{B}$ are *interdefinable* if for all $n \geq 1$ and all $X \subseteq A^n$, X is definable in $\mathcal{A}$ if and only if X is definable in $\mathcal{B}$.

Following [7], we will write $\mathcal{A} =_{\text{df}} \mathcal{B}$ to mean that $\mathcal{A}$ and $\mathcal{B}$ are interdefinable.

Definition 1.8. Let P be a binary predicate. Let $\mathcal{B} = (B, H)$ be an $\mathcal{L}_{or}(P)$-structure such that B is an ordered field. Let $\rho : B \to B^2$ be definable in $\mathcal{B}$. Let $H \subseteq B^2$ be the interpretation of the binary predicate P in $\mathcal{B}$. For $(z_1, \ldots, z_n) \in H^n \subseteq B^{2n}$, let

$$\rho(|z|) := (\rho(|z_1|), \ldots, \rho(|z_n|))$$

where $|\cdot|$ denotes modulus.

Recall that an $\mathcal{L}$-theory T is said to be *model complete* if for all models $\mathcal{M}, \mathcal{N}$ of T such that $\mathcal{M} \subseteq \mathcal{N}$, we have $\mathcal{M} \preceq \mathcal{N}$. It can be shown that T is model complete if and only if every $\mathcal{L}$-formula is equivalent to an existential formula in T. (This fact is part of Exercise 3.4.12 in [21].) An $\mathcal{L}$-theory T is said to be *near model complete* if every $\mathcal{L}$-formula is equivalent in T to a Boolean combination of existential formulas. This motivates the next definition.

Definition 1.9 (Predicate-near model completeness)**.** Let P be a binary predicate and let U be a unary predicate. Let $\mathcal{L}_{or}(P) := \mathcal{L}_{or} \cup \{P\}$ and let $\mathcal{L}_{or}(U) := \mathcal{L}_{or} \cup \{U\}$.

1. An $\mathcal{L}_{or}(U)$-structure $\mathcal{A}$ will be said to have PNMC if for all $m \geq 1$, every subset of A^m definable in $\mathcal{A}$

is a Boolean combination of subsets of A^m defined by formulas of the form

$$\exists x_1 \dots \exists x_n (\bigwedge_{i=1}^{n} (x_i \in U) \wedge \phi(x_1, \dots, x_n, v_1, \dots, v_m))$$

where ϕ is a quantifier free $\mathcal{L}_{or}$-formula with parameters from A.

2. An $\mathcal{L}_{or}(P)$-structure $\mathcal{A}$ will be said to have *predicate-near model completeness* (PNMC) if for all $m \geq 1$, every subset of A^m definable in $\mathcal{A}$ is a Boolean combination of subsets of A^m defined by formulas of the form

$$\exists x_1 \exists y_1 \dots \exists x_n \exists y_n (\bigwedge_{i=1}^{n} (x_i, y_i) \in P \wedge \phi(x_1, y_1, \dots, x_n, y_n, v_1, \dots, v_m))$$

where ϕ is a quantifier free $\mathcal{L}_{or}$-formula with parameters from A.

3. Let $\mathcal{B} = (B, H)$ be an $\mathcal{L}_{or}(P)$-structure such that B is an ordered field. Let $\rho : B \to B^2$ be definable in $\mathcal{B}$. We say that $\mathcal{B}$ has ρ-PNMC if for all $m \geq 1$, every subset of B^m definable in $\mathcal{B}$ is a Boolean combination of subsets of B^m defined by formulas of the form

$$\exists x_1 \exists y_1 \dots \exists x_n \exists y_n (\bigwedge_{i=1}^{n} (x_i, y_i) \in P \wedge \phi(x_1, y_1, \dots, x_n, y_n, \rho(|(x_1, y_1)|), \dots, \rho\,|(x_n, y_n)|), v_1, \dots, v_m))$$

where $|\cdot|$ denotes modulus and ϕ is a quantifier free $\mathcal{L}_{or}$-formula with parameters from B.

From the theorems stated in Section 1.1, we see that many expansions of $\overline{\mathbb{R}}$ by a finitely generated subgroup of $\mathbb{C}^\times$ have PNMC.

In addition to proving results about the definable sets in expansions of $\overline{\mathbb{R}}$ by finitely generated subgroups of $\mathbb{C}^\times$, we also prove some results about the *open* definable sets in such structures. To make these results precise, we give the definition of the open core of an ordered structure. This definition was introduced by Miller and Speissegger in [22] and has been helpful in classifying expansions of $\overline{\mathbb{R}}$ by finitely generated subgroups.

Definition 1.10 (Open core)**.** Let $\mathcal{R} = (R, <, \dots)$ be an ordered structure. The *open core* of $\mathcal{R}$, denoted $\mathcal{R}^o$, is the structure $(R, (U))$, where U ranges over the open subsets of R^n (for any $n > 0$) that are definable in $\mathcal{R}$.

Some expansions that we consider define a certain type of closed set, a spiral together with the origin.

Definition 1.11 (Logarithmic spiral)**.** Let $\omega \in \mathbb{R}^{\neq 0}$. The logarithmic spiral S_ω is defined as

$$S_\omega := e^{(i+\omega)\mathbb{R}}.$$

The spiral S_ω is parameterized by $(x(t), y(t)) = (e^{\omega t}\cos(t), e^{\omega t}\sin(t))$.

1.2.1 Expansions by finitely generated subgroups of $\mathbb{R}^{>0}$

Consider a structure of the form $(\overline{\mathbb{R}}, G)$, where G is a finitely generated subgroup of $\mathbb{R}^{>0}$. If G is not dense in $\mathbb{R}^{>0}$, then G is discrete. In this case, there is $a > 0$ such that $G = a^{\mathbb{Z}}$. Thus, there are two possibilities for a finitely generated subgroup G of $\mathbb{R}^{>0}$: either G is dense in $\mathbb{R}^{>0}$ or it is of the form $a^{\mathbb{Z}}$. From Corollary 1.3 and Theorem 1.4, we obtain the following theorem.

Theorem 1.12. *Let G be a finitely generated subgroup of $\mathbb{R}^{>0}$. The structure $(\overline{\mathbb{R}}, G)$ has PNMC.*

1.2.2 Expansions by subgroups of $\mathbb{C}^\times$ generated by two complex numbers

Consider $\mathcal{R} := (\overline{\mathbb{R}}, (ae^{i\varphi})^{\mathbb{Z}}(be^{i\psi})^{\mathbb{Z}})$, where $a, b \in \mathbb{R}^{>0}$ and $\varphi, \psi \in \mathbb{R}$. That is, $\mathcal{R}$ is an expansion of the real field $\overline{\mathbb{R}}$ by a subgroup G of $\mathbb{C}^\times$ generated by two complex numbers. Throughout this section, let $\Gamma = (ae^{i\varphi})^{\mathbb{Z}}(be^{i\psi})^{\mathbb{Z}}$.

Fig. 1.1 summarizes what we know about the definable sets in expansions of this form. The yellow cells denote results proved by the author. The proofs of these results are given in Chapter 5 and Chapter 6. The results in cells without citations are proved in Appendix A.

In Fig. 1.2, we summarize known results about the open core of structures of the form $(\overline{\mathbb{R}}, (ae^{i\varphi})^{\mathbb{Z}}(be^{i\psi})^{\mathbb{Z}})$. Consider $\mathbb{R}$ as a structure in the empty language. It is easy to see that $\mathbb{R}^o =_{\text{df}} \mathbb{R}$. Since the graphs of addition and multiplication are closed in $\mathbb{R}^3$ and the set $\{(x, y) \in \mathbb{R}^2 : x < y\}$ is open in $\mathbb{R}^2$, we also have $\overline{\mathbb{R}}^o =_{\text{df}} \overline{\mathbb{R}}$. (In fact, we have something even stronger: for all $m \geq 1$ and all $X \subseteq \mathbb{R}^m$ definable in $\overline{\mathbb{R}}$, X is a Boolean combination of open sets definable in $\overline{\mathbb{R}}$.)

Since $a^{\mathbb{Z}} \cup \{0\}$ is closed, the fact that $\overline{\mathbb{R}}^o =_{\text{df}} \overline{\mathbb{R}}$ together with Corollary 1.3 also gives us that $(\overline{\mathbb{R}}, a^{\mathbb{Z}})^o =_{\text{df}} (\overline{\mathbb{R}}, a^{\mathbb{Z}})$.

	$\varphi, \psi \in \pi\mathbb{Q}$	$\varphi \in \pi\mathbb{Q}, \psi \notin \pi\mathbb{Q}$	$\varphi \notin \pi\mathbb{Q}, \psi \in \pi\mathbb{Q}$	$\varphi \notin \pi\mathbb{Q}, \psi \notin \pi\mathbb{Q}$
$a = 1, b = 1$	$\mathcal{R} =_{\text{df}} \overline{\mathbb{R}}$	Γ is dense in $\mathbb{S}^1$; $(\overline{\mathbb{R}}, \Gamma)$ has PNMC [2]	$(\overline{\mathbb{R}}, \Gamma)$ has PNMC [2]	$(\overline{\mathbb{R}}, \Gamma)$ has PNMC [2]
$a = 1, b \neq 1$	$\mathcal{R} =_{\text{df}} (\overline{\mathbb{R}}, b^{\mathbb{Z}})$	$\mathcal{R} =_{\text{df}} (\overline{\mathbb{R}}, (be^{i\psi})^{\mathbb{Z}})$; $\mathcal{R}$ defines $\mathbb{Z}$ [19]	$\psi = 0$: $\mathcal{R}$ has PNMC (Theorem 5.7)	Unknown
$a \neq 1, b \neq 1$ and $\frac{\ln(a)}{\ln(b)} \in \mathbb{Q}$	$\mathcal{R} =_{\text{df}} (\overline{\mathbb{R}}, a^{\mathbb{Z}})$	$\varphi = 0$: $\mathcal{R} =_{\text{df}} (\overline{\mathbb{R}}, b^{\mathbb{Z}}(e^{i\psi})^{\mathbb{Z}})$ (Theorem 6.19)	$\psi = 0$: $\mathcal{R} =_{\text{df}} (\overline{\mathbb{R}}, a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}})$ (Theorem 6.19)	Unknown
$a \neq 1, b \neq 1$ and $\frac{\ln(a)}{\ln(b)} \notin \mathbb{Q}$	$\mathcal{R} =_{\text{df}} (\overline{\mathbb{R}}, a^{\mathbb{Z}}b^{\mathbb{Z}})$	$\varphi = 0$: • Γ dense in $\mathbb{C}$: $\mathcal{R}$ has ρ-PNMC (Theorem 6.17) • Γ not dense in $\mathbb{C}$: $\mathcal{R}$ defines S_ω (Proposition 4.13)	$\psi = 0$: • Γ dense in $\mathbb{C}$: $\mathcal{R}$ has ρ-PNMC (Theorem 6.17) • Γ not dense in $\mathbb{C}$: $\mathcal{R}$ defines S_ω (Proposition 4.13)	Unknown

Figure 1.1: Interdefinability of expansions by subgroups generated by two elements

	$\varphi, \psi \in \pi\mathbb{Q}$	$\varphi \in \pi\mathbb{Q}, \psi \notin \pi\mathbb{Q}$	$\varphi \notin \pi\mathbb{Q}, \psi \in \pi\mathbb{Q}$	$\varphi \notin \pi\mathbb{Q}, \psi \notin \pi\mathbb{Q}$
$a = 1, b = 1$	$\mathcal{R}^o =_{\text{df}} \overline{\mathbb{R}}$	$\mathcal{R}^o =_{\text{df}} \overline{\mathbb{R}}$ [18]	$\mathcal{R}^o =_{\text{df}} \overline{\mathbb{R}}$ [18]	$\mathcal{R}^o =_{\text{df}} \overline{\mathbb{R}}$ [18]
$a = 1, b \neq 1$	$\mathcal{R}^o =_{\text{df}} (\overline{\mathbb{R}}, b^{\mathbb{Z}})$ [10]	all open sets of all arities are definable in $\mathcal{R}$ [19]	$\psi = 0$: $\mathcal{R}^o =_{\text{df}} (\overline{\mathbb{R}}, b^{\mathbb{Z}})$ (Theorem 5.11)	Unknown
$a \neq 1, b \neq 1$ and $\frac{\ln(a)}{\ln(b)} \in \mathbb{Q}$	$\mathcal{R}^o =_{\text{df}} (\overline{\mathbb{R}}, a^{\mathbb{Z}})$	$\varphi = 0$: $\mathcal{R}^o =_{\text{df}} (\overline{\mathbb{R}}, b^{\mathbb{Z}})$ (Theorem 5.11)	$\psi = 0$: $\mathcal{R}^o =_{\text{df}} (\overline{\mathbb{R}}, a^{\mathbb{Z}})$ (Theorem 5.11)	Unknown
$a \neq 1, b \neq 1$ and $\frac{\ln(a)}{\ln(b)} \notin \mathbb{Q}$	$\mathcal{R}^o =_{\text{df}} \overline{\mathbb{R}}$ [4]	$\varphi = 0$, Γ dense in $\mathbb{C}$: $\mathcal{R}^o =_{\text{df}} (\overline{\mathbb{R}}, a^{\mathbb{Z}})$ (Theorem 6.18)	$\psi = 0$, Γ dense in $\mathbb{C}$: $\mathcal{R}^o =_{\text{df}} (\overline{\mathbb{R}}, b^{\mathbb{Z}})$ (Theorem 6.18)	Unknown

Figure 1.2: The open core of expansions by subgroups generated by two elements

1.3 New results

The main results of this thesis concern the definable sets in two expansions of $\overline{\mathbb{R}}$ by finitely generated subgroups of $\mathbb{C}$. Let $a, b \in \mathbb{R}$ with $a, b > 1$ and let $\varphi \in \mathbb{R} \setminus \pi\mathbb{Q}$.

In Chapter 5 we study $(\overline{\mathbb{R}}, a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}})$ and prove the following results.

Theorem A. *$(\overline{\mathbb{R}}, a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}})$ has PNMC.*

From Theorem A, it follows that $(\overline{\mathbb{R}}, a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}})$ does not satisfy any of (1)-(3) in Theorem 1.6. First note that $(\overline{\mathbb{R}}, a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}})$ defines both $(e^{i\varphi})^{\mathbb{Z}}$ and $a^{\mathbb{Z}}$ (as $a^{\mathbb{Z}} = \{|z| : z \in a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}}\}$). If $(\overline{\mathbb{R}}, a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}})$ defines $\mathbb{Z}$, then by [20, (37.6)], $(\overline{\mathbb{R}}, a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}})$ defines every Borel subset of $\mathbb{R}$. However, by Theorem A, every subset of $\mathbb{R}$ which is definable in $(\overline{\mathbb{R}}, a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}})$ is a Boolean combination of F_σ sets. The projection P of $a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}}$ onto the real line is a definable set that is dense and codense in $\mathbb{R}$. Since P is codense in $\mathbb{R}$, P has empty interior. Since P is dense in $\mathbb{R}$, P cannot be a finite union of discrete sets. Therefore, $(\overline{\mathbb{R}}, a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}})$ is not d-minimal. Lastly, the complement of $a^{\mathbb{Z}}$ in $\mathbb{R}^{>0}$ is open and definable in $(\overline{\mathbb{R}}, a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}})$. However, $\mathbb{R}^{>0} \setminus a^{\mathbb{Z}}$ is an infinite union of disjoint open intervals, and so not every open set definable in $(\overline{\mathbb{R}}, a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}})$ is semialgebraic.

Theorem B. *$(\overline{\mathbb{R}}, a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}})^o =_{df} (\overline{\mathbb{R}}, a^{\mathbb{Z}})$.*

In Chapter 6 we study $(\overline{\mathbb{R}}, (ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}})$. We first assume that the group $(ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}}$ is dense in $\mathbb{C}$. Let $\rho : \mathbb{R} \to \mathbb{R}^2$ be the function given by

$$\rho(x) = \begin{cases} (e^{i\varphi})^k, & x = a^k b^l \\ 1, & x \notin a^{\mathbb{Z}} b^{\mathbb{Z}}. \end{cases}$$

Theorem C. *Suppose that $(ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}}$ is dense in $\mathbb{C}$. Then $(\overline{\mathbb{R}}, (ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}})$ has ρ-PNMC.*

We also study what the open core of $(\overline{\mathbb{R}}, (ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}})$ is, whether $(ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}}$ is dense in $\mathbb{C}$ or not.

Theorem D. *Let $H = (ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}}$, where $a, b \neq 1$. Suppose that for all $\omega \in \mathbb{R}^\times$, $(\overline{\mathbb{R}}, H)$ does not define S_ω. Then exactly one of the following holds:*

1. *$(\overline{\mathbb{R}}, H)^o =_{df} \overline{\mathbb{R}}$.*
2. *$(\overline{\mathbb{R}}, H)^o =_{df} (\overline{\mathbb{R}}, b^{\mathbb{Z}})$.*

Chapter 2

Algebraic definitions and lemmas

In this section, we establish some algebraic facts about groups and fields which we will use to axiomatize the theories of the structures in Chapter 5 and Chapter 6.

2.1 Abelian groups

Since we consider multiplicative subgroups of fields, all groups we consider are abelian. Here we give some definitions and theorems about abelian groups which we will use later.

Definition 2.1. Let B be a subgroup of A. We say that B is *pure in* A, or that B is a pure subgroup of A, if $B \cap A^{[m]} = B^{[m]}$ for all $m \geq 1$. That is, an element of B has an mth root in A if and only if it has an mth root in B.

Let $B \subseteq A$ be a subgroup and let $S \subseteq A$. We define $B\langle S\rangle_A$ to be the subgroup of A given by

$$B\langle S\rangle_A := \Big\{a \in A : a^n = a' s_1^{k_1} \dots s_m^{k_m}, a' \in B, s_1, \dots, s_m \in S, k_1, \dots, k_m \in \mathbb{Z},$$

$$m \geq 0, n > 0\}.$$

If $S = \{a\}$ for some $a \in A \setminus B$, we write $B\langle a\rangle_A$ instead of $B\langle S\rangle_A$.

Lemma 2.2. *$B\langle S\rangle_A$ is a pure subgroup of A.*

Proof. Let x be an element of $B\langle S\rangle_A$ with an mth root y in A. Then there are $a' \in B$, $s_1, \dots, s_l \in S$, $k_1, \dots, k_l \in \mathbb{Z}$, and $n > 0$ such that

$$x^n = a' s_1^{k_1} \dots s_l^{k_l} = y^{mn}.$$

By definition of $B\langle S\rangle_A$, we also have $y \in B\langle S\rangle_A$. Therefore, x also has an mth root in $B\langle S\rangle_A$. $\square$

It is not hard to show that

$$B\langle a_1, \dots, a_n\rangle_A = (\dots(B\langle a_1\rangle_A)\langle a_2\rangle_A)\dots)\langle a_n\rangle_A$$

for $a_1, \ldots, a_n \in A$.

2.1.1 The rank of an abelian group

We now collect a few facts about abelian groups with finite rank. Recall that an abelian group Γ is said to have *finite rank* if Γ has a finitely generated subgroup Γ' such that Γ/Γ' is torsion. Every finitely generated abelian group has finite rank. However, not every group with finite rank is finitely generated. For example, let $\mathbb{U}$ be the group consisting of all roots of unity in $\mathbb{C}$. Every element of $\mathbb{U}$ has finite order, but $\mathbb{U}$ is not finitely generated.

Since we consider expansions of $\overline{\mathbb{R}}$ by finitely generated subgroups of $\mathbb{C}$, the following theorem is very helpful for us. This theorem is an immediate corollary of Theorem 1.1 of Evertse, Schlickewei, and Schmidt [16].

Theorem 2.3. *Every finite rank multiplicative subgroup of a field of characteristic 0 has the Mann property.*

We will also make use of the following theorem about finite rank subgroups of $\mathbb{C}^\times$, which is proved in [2].

Theorem 2.4 (Proposition 1.1, [2])**.** *Let Γ be a finite rank subgroup of $\mathbb{C}^\times$. For each $n > 0$, $[n]\Gamma$ is finite.*

2.2 Freeness of fields

Let E and F be field extensions of a field k, where E, F are subfields of a field K. We say that E and F are *free over* k if any set $S \subseteq E$ which is algebraically independent over k is also algebraically independent over F. Equivalently, E and F are free over k if any $S \subseteq E$ which is algebraically dependent over F is algebraically dependent over k.

The next fact is part of Proposition 12 in Section 14, Chapter V of [3].

Fact 2.5 (Proposition V.14.12, [3])**.** Let E and F be field extensions of a field k, where E, F are subfields of a field K. Then E and F are free over k if and only if there exists a transcendence basis of E over k which is algebraically independent over F.

The next fact follows easily from the definition of freeness. This fact is also part of Exercise 14 in Section 14, Chapter V in [3].

Fact 2.6 (Exercise V.14.14, [3])**.** Let E, F, G be three extensions of a field k contained in a field K such that $F \subseteq G$. If E and F are free over k and $E(F)$ and G are free over F, then E and G are free over k.

We will use the following lemma repeatedly in Chapter 5 and Chapter 6 to establish our axiomatizations of the theories of those structures.

Lemma 2.7. *Let K be a real closed field and let $A \leq K^\times$ and $G \lneqq \mathbb{S}^1(K)$. Let K' be a subfield of K, G' a subgroup of G, and A' a subgroup of A such that $G'A' \subseteq K'(i)$ and $K'(i)$ and $\mathbb{Q}(GA)$ are free over $\mathbb{Q}(G'A')$. Let E be a subset of G or of A, and let $X \subseteq K$ be a subset that is algebraically independent over $K'(GA)$. Then $K'(Re(E), X)^{rc}(i)$ and $\mathbb{Q}(GA)$ are free over $\mathbb{Q}(G'A', E)$.*

Proof. Since $K'(i)$ and $\mathbb{Q}(GA)$ are free over $\mathbb{Q}(G'A')$, we have $G' \subseteq K'(i)$ and $A' \subseteq K'$. If $E \subseteq G$, then for any $z \in E$, we have $\mathrm{Re}(z) = \frac{z^2+1}{2z}$ since $G \subseteq \mathbb{S}^1(K)$. Thus, $\mathrm{Re}(E) \subseteq \mathbb{Q}(E)$. If $E \subseteq A$, then $\mathrm{Re}(E) = E$ since $A \subseteq K$. Therefore, in both cases, we have $\mathrm{Re}(E) \subseteq \mathbb{Q}(E)$.

We claim that

$$K'(E, X) \text{ and } \mathbb{Q}(GA) \text{ are free over } \mathbb{Q}(G'A', E).$$

Suppose that we can prove this claim. We now show how it follows from the claim that $K'(\mathrm{Re}(E), X)^{\mathrm{rc}}(i)$ and $\mathbb{Q}(GA)$ are free over $\mathbb{Q}(G'A', E)$. By Fact 2.5, to prove freeness, it suffices to show that there exists a transcendence basis of $K'(\mathrm{Re}(E), X)^{\mathrm{rc}}(i)$ over $\mathbb{Q}(G'A', E)$ which is algebraically independent over $\mathbb{Q}(GA)$. Since we assume that $K'(E, X)$ and $\mathbb{Q}(GA)$ are free over $\mathbb{Q}(G'A', E)$, it also follows from Fact 2.5 that there is a transcendence basis Y of $K'(E, X)$ over $\mathbb{Q}(G'A', E)$ which is algebraically independent over $\mathbb{Q}(GA)$. We claim that Y is also a transcendence basis of $K'(\mathrm{Re}(E), X)^{\mathrm{rc}}(i)$ over $\mathbb{Q}(G'A', E)$.

To prove this, we must show in particular that $K'(\mathrm{Re}(E), X)^{\mathrm{rc}}(i)$ is an algebraic extension of $\mathbb{Q}(G'A', E)(Y)$. Note that

$$K'(\mathrm{Re}(E), X) \subseteq K'(E, X) \subseteq K'(\mathrm{Re}(E), X)^{\mathrm{rc}}(i)$$

so since $Y \subseteq K'(E, X)$, we also have $Y \subseteq K'(\mathrm{Re}(E), X)^{\mathrm{rc}}(i)$. From the above inclusions, we see that $K'(\mathrm{Re}(E), X)^{\mathrm{rc}}(i)$ is an algebraic extension of $K'(E, X)$. But by assumption, $K'(E, X)$ is an algebraic extension of $\mathbb{Q}(G'A', E)(Y)$. Therefore, $K'(\mathrm{Re}(E), X)^{\mathrm{rc}}(i)$ is an algebraic extension of $K'(E, X)$.

We now prove the claim.

By Fact 2.6, to prove the claim, it suffices to show that:

1. $\mathbb{Q}(GA)$ and $K'(E)$ are free over $\mathbb{Q}(G'A', E)$, and

2. $K'(E, X)$ and $\mathbb{Q}(GA)(K'(E))$ are free over $K'(E)$.

To prove (1), let $S \subseteq \mathbb{Q}(GA)$ be algebraically dependent over $K'(E)$. Then $S \cup E$ is a subset of $\mathbb{Q}(GA)$ which is algebraically dependent over $K'(i)$. By our assumption that $K'(i)$ and $\mathbb{Q}(GA)$ are free over $\mathbb{Q}(G'A')$, $S \cup E$ is also algebraically dependent over $\mathbb{Q}(G'A')$. Thus, S is algebraically dependent over $\mathbb{Q}(G'A', E)$.

For (2), first note that since we assume $E \subseteq G$ or $E \subseteq A$, $\mathbb{Q}(GA)(K'(E)) = K'(GA)$. It is clear that X is a transcendence basis of $K'(E, X)$ over $K'(E)$. Since we also assume that X is algebraically independent over $K'(GA)$, $K'(E, X)$ and $K'(GA)$ are free over $K'(E)$ by Fact 2.5. $\square$

Corollary 2.8. *Let G' be a subgroup of G and A' a subgroup of A. Let k be a subfield of K such that $k \subseteq \mathbb{Q}(Re(G'A'))$ and let $X \subseteq K$. Suppose that X is algebraically independent over $\mathbb{Q}(GA)$. Then $k(X)^{rc}(i)$ and $\mathbb{Q}(GA)$ are free over $\mathbb{Q}(G'A')$.*

Proof. First note that for $g \in G'$ and $a \in A'$, $\mathrm{Re}(ga) = \frac{(ga)^2+a^2}{2ga}$ since $G' \subseteq \mathbb{S}^1(K)$. Therefore, $\mathrm{Re}(G'A') \subseteq \mathbb{Q}(G'A')$. Since no (nonempty) subset of $k(i)$ is algebraically independent over $\mathbb{Q}(G'A')$, $k(i)$ and $\mathbb{Q}(GA)$ are free over $\mathbb{Q}(G'A')$. Note that $k(GA) \subseteq \mathbb{Q}(GA)$. Since we assume X is algebraically independent over $\mathbb{Q}(GA)$, X is also algebraically independent over $k(GA)$. Applying Lemma 2.7 with $E = \emptyset$, we see that $k(X)^{\mathrm{rc}}(i)$ and $\mathbb{Q}(GA)$ are free over $\mathbb{Q}(G'A')$. $\square$

Since $\emptyset$ is considered to be algebraically independent over any field, we will sometimes apply Lemma 2.7 and Corollary 2.8 with $X = \emptyset$.

2.3 Discrete ordered groups

Let K be an ordered field and let G be a subgroup of $K^\times$. Recall that G is said to be discrete if every point of G is isolated in the order topology. The structures we consider in this thesis are interdefinable with an expansion of $\overline{\mathbb{R}}$ of the form $(\overline{\mathbb{R}}, A, \ldots)$, where A is a discrete subgroup of $\mathbb{R}^{>0}$. Therefore, we need to prove some facts about discrete subgroups in ordered fields.

2.3.1 Regularly discrete groups

We start by stating some facts about general ordered abelian groups.

If A is an ordered abelian group, we say that $S \subseteq A$ is *convex* if for all $a, b \in S$ and all $x \in A$ such that $a < x < b$, we have $x \in S$.

Let A be an ordered abelian group written multiplicatively. An element ε of A is said to be the *smallest positive element* of A if $\varepsilon > 1$ and for all $x \in A$ such that $x > 1$, $x \geq \varepsilon$.

Definition 2.9. An ordered abelian group A with a smallest positive element is said to be *regularly discrete* if for all $n \geq 1$ and every infinite convex set $S \subseteq A$, $S \cap A^{[n]} \neq \emptyset$.

The next lemma follows from the proof of Theorem 2.1 in Zakon [27].

Lemma 2.10. *Let A be an ordered abelian group with a smallest element larger than 1, denoted ε. The following are equivalent:*

1. *A is regularly discrete;*

2. *for all $n \geq 1$, $A/A^{[n]} = \{\varepsilon A^{[n]}, \varepsilon^2 A^{[n]}, \ldots, \varepsilon^n A^{[n]}\}$.*

In particular, an ordered abelian group A with a smallest positive element is regularly discrete if and only if $[n]A = n$ for all $n \geq 1$.

From this lemma, we obtain the following corollary.

Corollary 2.11. *Let A be an ordered abelian group with a smallest positive element ε. Then A is regularly discrete if and only if for all $n \geq 1$ and all $a, b \in A$ such that (a, b) has at least n elements, $A^{[n]} \cap (a, b) \neq \emptyset$.*

Proof. Suppose A is regularly discrete. Let $n \geq 1$ and let $a, b \in A$ be such that (a, b) has at least n elements. If there is $k \in \{1, \ldots, n\}$ such that $a\varepsilon^k \geq b$, then since ε is the smallest positive element of A, $(a, b) = \{a\varepsilon, a\varepsilon^2, \ldots, a\varepsilon^{k-1}\}$. This contradicts our assumption that (a, b) has at least n elements. Therefore, $a\varepsilon^j \in (a, b)$ for all $j \in \{1, \ldots, n\}$. Since A is regularly discrete, by Lemma 2.10, there is $k \in \{1, \ldots, n\}$ such that $a\varepsilon^k \in A^{[n]}$. Therefore, $A^{[n]} \cap (a, b) \neq \emptyset$.

Conversely, suppose that A is not regularly discrete. By Lemma 2.10, there are $n \geq 1$ and $a \in A$ such that a is not equivalent to any of $\varepsilon, \varepsilon^2, \ldots, \varepsilon^n$ modulo $A^{[n]}$. Consider the interval $(a, a\varepsilon^{n+1})$. Since ε is the smallest positive element of A, this interval consists of the n elements $a\varepsilon, \ldots, a\varepsilon^n$. If there is $x \in A^{[n]} \cap (a, a\varepsilon^{n+1})$, then there is $k \in \{1, \ldots, n\}$ such that $a \equiv \varepsilon^{-k} \mod A^{[n]}$. We also have $a \equiv \varepsilon^{n-k} \mod A^{[n]}$ for this k, contradicting our choice of a. So $A^{[n]} \cap (a, b) = \emptyset$. □

2.3.2 Subgroups of the multiplicative group of a real closed field

We now consider discrete multiplicative subgroups of real closed fields. In this subsection, K will be a real closed field and A a multiplicative subgroup of $K^{>0}$ with smallest positive element ε. In particular, A contains the subgroup $\varepsilon^{\mathbb{Z}}$.

Lemma 2.12. *Let K be a real closed field and $A \leq K^{>0}$. Suppose that A has a smallest positive element ε and for all $x \in K$, there is $a \in A$ such that $a \leq x < a\varepsilon$. Then A is regularly discrete.*

Proof. Let $n \geq 1$ and let $a, b \in A$ be such that (a, b) has n elements. By Corollary 2.11, to prove that A is regularly discrete, it suffices to show that $A^{[n]} \cap (a, b) \neq \emptyset$. Since ε is the smallest positive element of A and we assume that (a, b) has n elements, we have

$$a < a\varepsilon < a\varepsilon^2 < \ldots < a\varepsilon^n < b.$$

Since K is real closed and $a, b \in K^{>0}$, we also have $a^{1/n}, b^{1/n} \in K$. Moreover, the function $x \mapsto x^n$ is increasing on $K^{>0}$. By assumption, there is $c \in A$ such that $c \leq a^{1/n} < c\varepsilon$. Therefore, we also have $c^n < c^n\varepsilon^n \leq a\varepsilon^n < b$. Putting everything together, we have $a < (c\varepsilon)^n \leq a\varepsilon^n < b$. Since ε is the smallest positive element of A, we must have $(c\varepsilon)^n = a\varepsilon^j$ for some $j \in \{1, \ldots, n\}$. Since $c\varepsilon \in A$, we have $A^{[n]} \cap (a, b) \neq \emptyset$, as desired. $\square$

If $\varepsilon^{\mathbb{Z}}$ is cofinal in A, then we can say more about A.

Lemma 2.13. *Let K be a real closed field and $A \leq K^{>0}$ be a subgroup with smallest positive element ε such that for all $a \in A$, there is $l \in \mathbb{Z}$ such that $\varepsilon^l \leq a < \varepsilon^{l+1}$. Then $A = \varepsilon^{\mathbb{Z}}$.*

Proof. Suppose for a contradiction that there is $a \in A$ such that $a \neq \varepsilon^k$ for any $k \in \mathbb{Z}$. Since we assume that $A \subseteq K^{>0}$, we either have $a > 1$ or $0 < a < 1$. If $a > 1$, then there is $k \in \mathbb{N}$ such that $\varepsilon^k < a < \varepsilon^{k+1}$. Therefore, $1 < a\varepsilon^{-k} < \varepsilon$, and $a\varepsilon^{-k}$ is a positive element of A smaller than ε. This contradicts our definition of ε. If $a < 1$, then there is $k \in \mathbb{N}$ such that $\varepsilon^{-(k+1)} < a < \varepsilon^{-k}$. In this case, $a\varepsilon^{k+1}$ is a positive element of A smaller than ε. $\square$

In particular, if A is a discrete subgroup of $\mathbb{R}^{>0}$, then A has a smallest positive element ε. By the previous lemma, $A = \varepsilon^{\mathbb{Z}}$. The group $\varepsilon^{\mathbb{Z}}$ is also regularly discrete by Lemma 2.10.

The previous lemma does not apply to subgroups of real closed fields in general. Using compactness, we can find a real closed field K and $a \in K$ such that $a > n$ for all $n \geq 1$. Then $2^{\mathbb{Z}}a^{\mathbb{Z}}$ is a subgroup of $K^{>0}$ with smallest positive element 2.

Suppose now that for every $x \in K^{>0}$, there is $a \in A$ such that $a \leq x < a\varepsilon$. Following van den Dries [10], we define a function $\lambda : K \to A$ by

$$\lambda(x) = \begin{cases} 0, & x \leq 0 \\ a, & x > 0 \text{ and } a \leq x < a\varepsilon \end{cases}$$

Since ε is the smallest positive element of A, for every $x \in K^{>0}$, there is exactly one element $a \in A$ such that $a \leq x < a\varepsilon$. Therefore, λ is well-defined.

Definition 2.14. Let K, A, and ε be as above. We say that K is *closed under* λ if

$$\lambda(K^{>0}) = K^{>0} \cap A.$$

The following lemmas will be used in our axiomatizations in Chapter 5 and Chapter 6.

Lemma 2.15. *For $k_1, k_2 \in K^{>0}$, $\lambda(k_1k_2) = \lambda(k_1)\lambda(k_2)$ or $\lambda(k_1k_2) = \lambda(k_1)\lambda(k_2)\varepsilon$. In general, for $k_1, \ldots, k_n \in K^{>0}$, $\lambda(k_1 \ldots k_n) = \lambda(k_1) \ldots \lambda(k_n)\varepsilon^j$ for some $j \in \{0, \ldots, n-1\}$.*

Proof. Let $a_1 = \lambda(k_1)$, $a_2 = \lambda(k_2)$. Note that in particular, $a_1, a_2 > 0$. We have $a_1a_2 \leq k_1k_2 < a_1\varepsilon a_2\varepsilon = a_1a_2\varepsilon^2$. Moreover, $a_1a_2 < a_1a_2\varepsilon < a_1a_2\varepsilon^2$, so by definition of λ, we must have $\lambda(k_1k_2) = a_1a_2$ or $\lambda(k_1k_2) = a_1a_2\varepsilon$. The proof of the last statement is similar. ☐

Now let Γ be a subgroup of $\mathbb{S}^1$ and $\Delta = \varepsilon^{\mathbb{Z}}$ for some $\varepsilon \in \mathbb{R}$ with $\varepsilon > 1$. Suppose that $(K, (\gamma\delta)_{\gamma\in\Gamma,\delta\in\Delta})$ satisfies the orientation axioms for $\Gamma\Delta$. (For the definition of orientation axioms, see Section 3.3.) In this case, we can say more about how λ behaves on K. We will use the following lemmas in Chapter 5 and Chapter 6.

We say that an element of a real closed field K is finite if there is $n \in \mathbb{N}$ such that $|k| < n$. We begin by noting that since $(K, (\gamma\delta)_{\gamma\in\Gamma,\delta\in\Delta})$ satisfies the orientation axioms for $\Gamma\Delta$, ε is finite.

Lemma 2.16. *For all finite $x \in K^{>0}$, there is $l \in \mathbb{Z}$ such that $\varepsilon^l \leq x < \varepsilon^{l+1}$. Therefore, for each finite $x \in K^{>0}$, $\lambda(x) = \varepsilon^l$ for some $l \in \mathbb{Z}$.*

Proof. First note that $\varepsilon^{\mathbb{Z}}$ is cofinal in the set of finite elements of K. This is because $\varepsilon^{\mathbb{Z}}$ is cofinal in $\mathbb{N} \subseteq \mathbb{R}$, and $(K, (\gamma\delta)_{\gamma\in\Gamma,\delta\in\Delta})$ satisfies the orientation axioms for $\Gamma\Delta$ by assumption. If $y \in K^{>0}$ is finite and $y > 1$, let l be the smallest natural number such that $y < \varepsilon^{l+1}$. If $0 < y \leq 1$, let m be the smallest natural number such that $y < \varepsilon^{-m+1}$, and then take $l = -m$. ☐

Lemma 2.17. *Let $K' = \mathbb{Q}(Re(\Gamma\Delta))^{rc}$. Then $\lambda((K')^{>0}) = \Delta$.*

Proof. It follows from the orientation axioms that every positive element of K' is finite. By Lemma 2.16, $\lambda((K')^{>0}) = \Delta$. ☐

Lemma 2.18. *Δ is a pure subgroup of A.*

Proof. Let $a \in A$ be such that $a^n \in \Delta$ for some $n > 0$. Since $a^n \in \Delta$, there is $l \in \mathbb{Z}$ such that $a^n = \varepsilon^l$. If $a \geq 1$, then since $A \subseteq K^{>0}$, we have $0 < a \leq a^n = \varepsilon^l$. In particular, a is finite. If $0 < a < 1$, then clearly a is finite.

Suppose for a contradiction that $a \notin \Delta$. Then since a is finite, by Lemma 2.16, there is $k \in \mathbb{Z}$ such that $\varepsilon^k < a < \varepsilon^{k+1}$. But then $1 < a\varepsilon^{-k} < \varepsilon$, contradicting our assumption that ε is the smallest element of A larger than 1. So we must have $a \in \Delta$. ☐

Lastly, we recall the following facts about subgroups of $\mathbb{R}^{>0}$.

Fact 2.19 (Exercise 1.3.14, [1])**.** Let G be an infinite additive subgroup of $\mathbb{R}$. Then G is dense in $\mathbb{R}$ or has a smallest element larger than 0.

Proof. Let $a = \inf\{g \in G : g > 0\}$. If $a \in G$, then a is the smallest element of G larger than 0. It can be shown that if $a \notin G$, then G is dense in $\mathbb{R}$. $\square$

Corollary 2.20. *Let G be an infinite multiplicative subgroup of $\mathbb{R}^{>0}$. If G is not dense in $\mathbb{R}^{>0}$, then G is regularly discrete.*

2.4 Oriented groups

Oriented groups were introduced by Günaydın in [17]. We refer the reader to Section 8.1 of [17] for the precise definition of an oriented group G with orientation $\mathcal{O}$. In this thesis, we will only consider one particular orientation: an orientation on on $\mathbb{S}^1(K)$, where K is a real closed field.

Let $\mathcal{L}_{orm}$ be the language of oriented monoids; that is, $\mathcal{L}_{orm} = \{\mathcal{O}, 1, \cdot\}$, where $\mathcal{O}$ is a ternary relation. Let K be a real closed field and G a multiplicative subgroup of $\mathbb{S}^1(K) \subseteq K^2$. Thus, the identity of G is the element $(1, 0)$ of K^2 and multiplication on G is defined by

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2).$$

where multiplication and addition in the components on the right side are performed in K. As in [17], we can make G into an oriented subgroup by taking the orientation $\mathcal{O}$ on G to be the one inherited from $\mathbb{S}^1(K)$.

In order to define the orientation on $\mathbb{S}^1(K)$, we first define an orientation $\mathcal{O}_{\mathbb{R}}$ on $\mathbb{S}^1$. For $z \in \mathbb{S}^1$, let $t(z)$ be the unique element of $[0, 1)$ such that $z = (\cos(2\pi t(z)), \sin(2\pi t(z))$. For $x, y \in \mathbb{R}$, we say $x \equiv y \mod 1$ if $x - y \in \mathbb{Z}$.

Let $\sigma_1, \sigma_2, \sigma_3 : \{1, 2, 3\} \to \{1, 2, 3\}$ be the permutations defined by

$$\sigma_1 := \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \sigma_2 := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \sigma_3 := \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Definition 2.21. For $z_1, z_2, z_3 \in \mathbb{S}^1$, we define $\mathcal{O}_{\mathbb{R}}$ by taking

$$\begin{aligned} \mathcal{O}_{\mathbb{R}}(z_1, z_2, z_3) \leftrightarrow{} & \text{there are } t_1, t_2, t_3 \in \mathbb{R} \text{ and } j \in \{1, 2, 3\} \text{ such that } t_1 < t_2 < t_3,\ t_3 - t_1 < 1, \\ & \text{and } t(z_{\sigma_j(i)}) \equiv t_i \mod 1 \text{ for } i = 1, 2, 3. \end{aligned}$$

This orientation is pictured below.

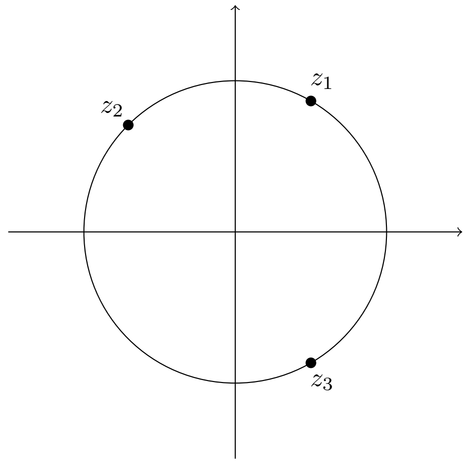

Figure 2.1: $\mathcal{O}_{\mathbb{R}}(z_1, z_2, z_3)$, $\mathcal{O}_{\mathbb{R}}(z_2, z_3, z_1)$, and $\mathcal{O}_{\mathbb{R}}(z_3, z_1, z_2)$ all hold.

As discussed in the example in Section 8.1 of [17], there is a quantifier free $\mathcal{L}_{or}$-formula $\varphi(x_1, y_1, x_2, y_2, x_3, y_3)$ such that for all $a_1, b_1, a_2, b_2, a_3, b_3 \in \mathbb{R}$,

$$\mathcal{O}_{\mathbb{R}}((a_1, b_1), (a_2, b_2), (a_3, b_3)) \leftrightarrow \overline{\mathbb{R}} \models \varphi(a_1, b_1, a_2, b_2, a_3, b_3).$$

Now let K be an arbitrary real closed field. We define an orientation $\mathcal{O}_K$ on $\mathbb{S}^1(K)$ by taking

$$\mathcal{O}_K((a_1, b_1), (a_2, b_2), (a_3, b_3)) \leftrightarrow K \models \varphi(a_1, b_1, a_2, b_2, a_3, b_3).$$

for $a_1, b_1, a_2, b_2, a_3, b_3 \in K$.

If the underlying real closed field K is clear from context, then we will write $\mathcal{O}$ instead of $\mathcal{O}_K$.

We say that G is *dense* in $\mathbb{S}^1(K)$ if for all $a, b \in \mathbb{S}^1(K)$ such that $\mathcal{O}(1, a, b)$ holds, there is $g \in G$ with $\mathcal{O}(a, g, b)$. We say that G is *regularly dense* in $\mathbb{S}^1(K)$ if for all $a, b \in \mathbb{S}^1(K)$ such that $\mathcal{O}(1, a, b)$ holds and for all $n \geq 1$, there is $g \in G$ with $\mathcal{O}(a, g^n, b)$.

Note if G is dense in $\mathbb{S}^1(K)$, then G is also regularly dense in $\mathbb{S}^1(K)$. To see this, let $a, b \in \mathbb{S}^1(K)$ be such that $\mathcal{O}(1, a, b)$ holds and let $n \geq 1$. Since K is real closed, there are $\alpha, \beta \in \mathbb{S}^1(K)$ such that $\alpha^n = a$, $\beta^n = b$, and $\mathcal{O}(1, \alpha, \beta)$ holds. Since G is dense, there is $g \in G$ such that $\mathcal{O}(\alpha, g, \beta)$ also holds. Again, since K is real closed, we also have $\mathcal{O}(a, g^n, b)$.

Let $\Gamma \subseteq G$ be an infinite subgroup. Let $x = (x_1, \ldots, x_n)$ be a tuple of n variables, and let $z = ((z_{11}, z_{12}), \ldots, (z_{n1}, z_{n2}))$ be a tuple of n pairs of variables. Let $\phi(x)$ be an $\mathcal{L}_{orm}(\Gamma)$-formula. From the definition of multiplication, orientation, and identity in G, we see that there is an $\mathcal{L}_{or}(\Gamma)$-formula $\psi_\phi(z)$ such that for all $(a_1, \ldots, a_n) \in G^n$ (with $a_i = (a_{i1}, a_{i2})$ for some $a_{i1}, a_{i2} \in K$),

$$(G, \mathcal{O}, 1, \cdot) \models \phi(a_1, \ldots, a_n) \text{ if and only if}$$

$$(K, <, +, -, 0, 1, \cdot) \models \psi_\phi((a_{11}, a_{12}), \ldots, (a_{n1}, a_{n2})).$$

In particular, all quantifiers that appear in ψ_ϕ must appear in pairs of $\exists$ or $\forall$, and there must be an even number of free variables in ψ_ϕ.

Let $\Sigma_{\mathrm{orm}}(\Gamma) := \{\psi_\phi : \phi \text{ an } \mathcal{L}_{orm}(\Gamma)\text{-formula}\}$. Note that $\Sigma_{\mathrm{orm}}(\Gamma)$ is closed under conjunctions and disjunctions of formulas, negation, and quantification over a pair of variables.

Definition 2.22. Let P be a binary predicate. The P-restriction θ_P of $\theta \in \Sigma_{\mathrm{orm}}(\Gamma)$ is defined recursively in analogy with the U-restriction defined on page 10 of [17]. In the following, θ, θ', and θ'' are formulas in $\Sigma_{\mathrm{orm}}(\Gamma)$.

- If θ is an atomic $\mathcal{L}_{or}(\Gamma)$-formula, then $\theta_P := \theta$;
- if $\theta = \neg\theta'$, then $\theta_P := \neg\theta'_P$;
- if $\theta = \theta' \wedge \theta''$, then $\theta_P := \theta'_P \wedge \theta''_P$;
- if $\theta = \theta' \vee \theta''$, then $\theta_P := \theta'_P \vee \theta''_P$;
- if $\theta = \exists x_1 \exists y_1 \theta'$, then $\theta_P := \exists x_1 \exists y_1 (P(x_1, y_1) \wedge \theta'_P)$;
- if $\theta = \forall x_1 \forall y_1 \theta'$, then $\theta_P := \forall x_1 \forall y_1 (P(x_1, y_1) \to \theta'_P)$.

We give the precise definition of V-restriction (where V is a unary predicate) in Section 5.2.1.

The following lemma is proved in Section 8.1.2 of [17]. We will use this lemma in the proof of Theorem 5.1 and to prove quantifier elimination in Section 3.4.

Lemma 2.23. *Let A and B be regularly dense oriented abelian groups such that $[p]A = [p]B$ for every prime p. Let A' and B' be pure subgroups of A and B respectively such that $A'_{tor} = A_{tor}$ and $B'_{tor} = B_{tor}$. Let $f : A' \to B'$ be an oriented abelian group isomorphism. Suppose that B is κ-saturated, where $\kappa > |B'|$ is uncountable, and let $a \in A \setminus A'$. There is $b \in B$ such that there is an oriented group isomorphism $A'\langle a\rangle_A \to B'\langle b\rangle_B$ extending f which takes a to b.*

2.5 The Archimedean valuation

We will need a few tools from valuation theory in Chapter 5 and Chapter 6. The most important tool is a theorem which we call the Fundamental Lemma, which allows us to describe how the function λ defined in Section 2.3.2 interacts with field extensions of a real closed field. We refer the reader to [9] for the basic definitions from valuation theory needed for this subsection.

Let K be an ordered field and let $K^\times$ denote the nonzero elements of K. We define an equivalence relation $\sim$ on $K^\times$ by setting $x \sim y$ (for $x, y \in K^\times$) if and only if there is $n > 1$ such that

$$\frac{1}{n} < \left|\frac{y}{x}\right| < n.$$

It is easy to see that $\sim$ is an equivalence relation. Denote the natural quotient map from $K^\times$ to $K^\times/{\sim}$ by v. For $x, y \in K^\times$, we define an operation $+ : v(K^\times) \times v(K^\times) \to v(K^\times)$ by $v(x) + v(y) := v(xy)$. For $x, y \in K^\times$, we define an ordering on $v(K^\times)$ by taking $v(x) > v(y)$ if and only if $\left|\frac{x}{y}\right| < \frac{1}{n}$ for all $n \geq 1$. It can be shown that these definitions do not depend on our choice of representative, and $(K^\times/{\sim}, +, <)$ is an ordered abelian group.

Lemma 2.24. *The map v is a valuation on $K^\times$.*

Proof. We must check that for all $x, y \in K^\times$,

1. $v(x+y) \geq \min\{v(x), v(y)\}$, and

2. $v(xy) = v(x) + v(y)$.

Note that (2) holds by definition of the operation on $v(K^\times)$.

Let $x, y \in K^\times$. To show that (1) holds, suppose without loss of generality that $v(y) \geq v(x)$. Then there is $m > 1$ such that $\left|\frac{y}{x}\right| < m$. If there is a positive integer N such that $\frac{1}{N} < \left|1 + \frac{y}{x}\right|$, then since we also have $\left|1 + \frac{y}{x}\right| < 1 + m$, $v(x+y) = v(x)$ by definition of v. Otherwise, if $\left|1 + \frac{y}{x}\right| < \frac{1}{N}$ for all positive integers N, then $v(x+y) > v(x)$ by definition of the ordering on $K^\times/\sim$. Therefore, $v(x+y) \geq \min\{v(x), v(y)\}$. $\square$

We call v the *Archimedean valuation* on K. Throughout, when we refer to a valuation v on an ordered field K, v is the Archimedean valuation on $K^\times$.

We will need the idea of an RCF-convex subring of a real closed field K to apply an important lemma for the Fundamental Lemma. We define an RCF-convex subring to be a T-convex subring as defined in [14] when $T = \mathrm{RCF}$. That is, an RCF-convex subring of a real closed field K is a convex subring $V \subseteq K$ such that $f(V) \subseteq V$ for each function $f : K \to K$ 0-definable in K. Let $\mathrm{RCF}_{\mathrm{convex}}$ be the theory of pairs (K, V), where K is a real closed field and V is a proper RCF-convex subring, axiomatized in the language $\mathcal{L}_{or}$ with an extra unary relation symbol for V. We recall the following fact about convex subrings of real closed fields from [14].

Fact 2.25 ((2.8), [14])**.** In a real closed field K, every convex subring of K is RCF-convex.

Next we prove several facts about the Archimedean valuation. We first recall some definitions from [9].

Definition 2.26. Let K be a field, and let w be a valuation on $K^\times$ with value group Γ. Let 0 be the identity of Γ, and let

$$\mathcal{O}_w := \{x \in K : w(x) \geq 0\}, \mathfrak{m}_w := \{x \in K : w(x) > 0\}.$$

It can be shown that $\mathcal{O}_w$ is a valuation ring with maximal ideal $\mathfrak{m}_w$.

Lemma 2.27. *Let K be an ordered field and let v be the Archimedean valuation. We have*

$$\mathcal{O}_v = \{x \in K : |x| < n \textit{ for some } n \geq 1\}$$

and

$$\mathfrak{m}_v = \{x \in K : |x| < \frac{1}{n} \textit{ for all } n \geq 1\}.$$

Moreover, $\mathcal{O}_v$ is convex.

Proof. By definition of v, $v(y) = 0$ if and only if there is $n \geq 1$ such that $\frac{1}{n} < |y| < n$. By definition of the ordering on $K^\times/\sim$, $v(y) > 0$ if and only if $|y| < \frac{1}{n}$ for all $n \geq 1$ by definition of the ordering on $K^\times/\sim$. Therefore, $v(y) \geq 0$ if and only if there is $n \geq 1$ such that $|y| < n$.

Next we prove that $\mathcal{O}_v$ is convex. Let $x, y \in \mathcal{O}_v$ and let $z \in K$ be such that $x < z < y$. Depending on the position of x and y relative to 0 in the ordering, we either have $|z| < |y|$ or $|z| < |x|$. In either case, there is $n \in \mathbb{N}$ such that $|z| < n$ by our assumption that $x, y \in \mathcal{O}_v$. $\square$

Thus, if K is a real closed field, then $(K, \mathcal{O}_v) \models \mathrm{RCF}_{\mathrm{convex}}$.

If K is a real closed field, then we consider $v(K^\times)$ as a $\mathbb{Q}$-vector space as follows. We define scalar multiplication on $v(K^\times)$ by $q \cdot v(a) = v(|a|^q)$ for $a \in K^\times$ and $q \in \mathbb{Q}$. It can be shown that the addition and scalar multiplication operations are well-defined by definition of v. Moreover, these operations make $v(K^\times)$ into a $\mathbb{Q}$-vector space.

The following theorem is Corollary 5.6 in van den Dries [11], applied to fit our situation. Note that we can apply Corollary 5.6 because $(K, \mathcal{O}_v) \models \mathrm{RCF}_{\mathrm{convex}}$ whenever K is a real closed field. We call this theorem the valuation inequality.

Theorem 2.28 (Valuation inequality)**.** *Let K be a real closed field and let K' be a real closed subfield of K. Let $a \in K$ and let $\Gamma = v(K'(a)^{rc})$ and $\Gamma' = v(K\prime)$. As a $\mathbb{Q}$-vector space, we have $\dim_{\mathbb{Q}}(\Gamma/\Gamma') \leq 1$.*

In the rest of this subsection, let ε be a real number greater than 1 and let $\Delta = \varepsilon^{\mathbb{Z}}$. Let K be a real closed field and let A be a subgroup of $K^{>0}$ containing Δ such that for every $x \in K^{>0}$, there is $a \in A$ such that $a \leq x < a\varepsilon$. Take $\lambda : K \to A$ to be the function defined in Section 2.3.2 and let $\mu : K \to K(i)$ be any function such that $\mu(x) = 1$ for all $x \in A$. As usual, if $S \subseteq K(i)$, then by S^{rc} we mean $(\mathrm{Re}(S))^{\mathrm{rc}}$.

Lemma 2.29 (Fundamental Lemma)**.** *Let K' be a real closed subfield of K that contains Δ and is closed under λ. Let $A' = K' \cap A$ and let $b \in K$. Then one of the following holds:*

1. *$\lambda(K'(b, \mu(b))^{rc}) = \lambda(K')$, or*
2. *there is an $\mathcal{L}_{or}(K')$-definable function $\tau : K^3 \to K$ such that*

$$\lambda((K'(b, \mu(b))^{rc})^{>0}) = A'\langle\lambda(\tau(b, \mu(b)))\rangle_A.$$

If $b \in A$, then we can take $\tau = id$ in the second case.

Proof. Let $D = \dim_{\mathbb{Q}}(v(K'(b, \mu(b))^{\mathrm{rc}})/v(K'))$. As $\mathbb{Q}$-vector spaces,

$$(v(K'(b, \mu(b))^{\mathrm{rc}})/v(K')) \,/\, (v(K'(b)^{\mathrm{rc}})/v(K')) \cong v(K'(b, \mu(b))^{\mathrm{rc}})/v(K'(b)^{\mathrm{rc}}).$$

Therefore, by rank-nullity,

$$D = \dim_{\mathbb{Q}}(v(K'(b, \mu(b))^{\mathrm{rc}})/v(K'(b))^{\mathrm{rc}})) + \dim_{\mathbb{Q}}(v(K'(b)^{\mathrm{rc}})/v(K')).$$

Let $d_1 := \dim_{\mathbb{Q}}(v(K'(b, \mu(b))^{\mathrm{rc}})/v(K'(b))^{\mathrm{rc}})$ and $d_2 := \dim_{\mathbb{Q}}(v(K'(b)^{\mathrm{rc}})/v(K'))$. By the valuation inequality, $d_1 \leq 1$ and $d_2 \leq 1$. Therefore, we also have $D \leq 2$.

Let $V = v(K'(b, \mu(b))^{\mathrm{rc}})$, let $W = v(K'(b))$, and let $X = v(K')$. We must consider several cases based on the value of D.

Case 1: $D = 0$. In this case, we have $V = W = X$. We will show that $\lambda(K'(b, \mu(b))^{\mathrm{rc}}) = \lambda(K')$.

Let $x \in (K'(b, \mu(b))^{\mathrm{rc}})^{>0}$. By assumption, there is $y \in K'$ such that $v(x) = v(y)$. By definition of λ, for any $k \in K^{>0}$, $\lambda(k) \leq k < \lambda(k)\varepsilon$; therefore, $v(x) = v(\lambda(x))$ and $v(y) = v(\lambda(y))$. Since we assume that $v(x) = v(y)$, $\frac{\lambda(x)}{\lambda(y)}$ is finite. Therefore, we must actually have $\frac{\lambda(x)}{\lambda(y)} = \varepsilon^l$ for some $l \in \mathbb{Z}$ by Lemma 2.16. Since $\lambda(x), \lambda(y) \in A$, $\lambda(\lambda(x)\lambda(y)^{-1}) = \lambda(x)\lambda(y)^{-1}$. By our assumption that K' is closed under λ and contains Δ, we have $\lambda(y)\varepsilon^l \in A'$. Therefore, $\lambda(x) \in A'$. This proves that $\lambda(K'(b, \mu(b))^{\mathrm{rc}})) \subseteq \lambda(K')$. The other inclusion is clear.

Note that we have actually proved that if $v((K'(b, \mu(b))^{\mathrm{rc}})^{\times}) = v((K')^{\times})$, then $\lambda(K'(b, \mu(b))^{\mathrm{rc}}) = \lambda(K')$.

Case 2: $D = 2$. Then by the valuation inequality, we must have $d_1 = 1$ and $d_2 = 1$. Therefore, there is $z \in K'(b, \mu(b))^{\mathrm{rc}}$ such that $v(z) \notin v(K'(b)^{\mathrm{rc}})$ and $z > 0$. Similarly, there is $y \in K'(b)^{\mathrm{rc}}$ such that $v(y) \notin v(K')$ and $y > 0$.

Let $z = \sigma(b, \mu(b))$, where σ is an $\mathcal{L}_{or}(K')$-definable function. Let $y = \theta(b)$, where θ is an $\mathcal{L}_{or}(K')$-definable function.

In this case, $V \cong W \oplus V/W$. To see this, note that

$$0 \to W \to V \to V/W \to 0$$

is a short exact sequence, and every short exact sequence of vector spaces splits. Since $V/W = \{v(z) + W\}$, we have $V/W \cong \mathbb{Q}v(z)$, where $\mathbb{Q}v(z)$ is the $\mathbb{Q}$-vector space generated by $v(z)$. Also,

$$0 \to X \to W \to W/X \to 0$$

is a short exact sequence of vector spaces, so $W \cong X \oplus W/X$. Thus,

$$V = X \oplus \mathbb{Q}v(z) \oplus \mathbb{Q}v(y).$$

Let $x \in K'(b, \mu(b))^{\mathrm{rc}}$ be such that $x > 0$. By the above, there are $k \in K'$ and $q, r \in \mathbb{Q}$ such that $v(x) = v(k) + qv(z) + rv(y)$.

Let $q = \frac{a_1}{b_1}$ and let $r = \frac{a_2}{b_2}$, where $a_1, a_2, b_1, b_2 \in \mathbb{Z}$ and $b_1 \neq 0$, $b_2 \neq 0$. Then $b_1 b_2 v(x) = b_1 b_2 v(k) + a_1 b_2 v(z) + a_2 b_1 v(y)$. By definition of λ, $v(c) = v(\lambda(c))$ for any $c \in K^{\times}$. Therefore, there is $N \in \mathbb{N}$ such that

$$\frac{1}{N} < \frac{\lambda(x)^{b_1 b_2}}{\lambda(k)^{b_1 b_2} \lambda(z)^{a_1 b_2} \lambda(y)^{a_2 b_1}} < N.$$

Let $\alpha = \frac{\lambda(x)^{b_1 b_2}}{\lambda(k)^{b_1 b_2} \lambda(z)^{a_1 b_2} \lambda(y)^{a_2 b_1}}$. Since α is a finite element of A, there is $l \in \mathbb{Z}$ such that $\alpha = \varepsilon^l$. By Lemma 2.15, there is $j \in \mathbb{Z}$ such that

$$\lambda(z)^{a_1 b_2} \lambda(y)^{a_2 b_1} = \lambda(z^{a_1 b_2} y^{a_2 b_1}) \varepsilon^j.$$

Therefore,

$$\lambda(x)^{b_1b_2} = \lambda(k)^{b_1b_2}\lambda(z^{a_1b_2}y^{a_2b_1})\varepsilon^j.$$

Since we assume that K' is closed under λ, $\lambda(k) \in A'$. So $\lambda(x) \in A'\langle\lambda(z^{a_1b_2}y^{a_2b_1})\rangle_A$ by definition. But $z^{a_1b_2}y^{a_2b_1} = (\sigma(b,\mu(b)))^{a_1b_2} \cdot (\theta(b))^{a_2b_1}$, so there is an $\mathcal{L}_{or}(K')$-function τ such that

$$\lambda(x) \in A'\langle\lambda(\tau(b,\mu(b)))\rangle_A.$$

This proves that $\lambda((K'(b,\mu(b))^{\mathrm{rc}})^{>0}) \subseteq A'\langle\lambda(\tau(b,\mu(b)))\rangle_A$.

Now let $x \in A'\langle\lambda(\sigma(b,\mu(b))\rangle_A$, where σ is an $\mathcal{L}_{or}(K')$-definable function. In particular, $x \in A$, so $\lambda(x) = x$. Then there are $a \in A'$, $d > 0$, and $l \in \mathbb{Z}$ such that $x = (a\lambda(\sigma(b,\mu(b)))^l)^{1/d}$. Since $v(\sigma(b,\mu(b))) = v(\lambda(\sigma(b,\mu(b))))$, we have $v(\sigma(b,\mu(b))^l) = v(\lambda(\sigma(b,\mu(b)))^l)$. Therefore, there is $N_1 \in \mathbb{N}$ such that $\frac{1}{N_1} < \frac{(a'\sigma(b,\mu(b))^l)^{1/d}}{(a'\lambda(\sigma(b,\mu(b)))^l)^{1/d}} < N_1$. That is, $\frac{1}{N_1} < \frac{(a'\sigma(b,\mu(b))^l)^{1/d}}{x} < N_1$. Moreover, $v((a'\sigma(b,\mu(b))^l)^{1/d}) = v(\lambda((a'\sigma(b,\mu(b))^l)^{1/d}))$, so there is $N_2 \in \mathbb{N}$ such that $\frac{1}{N_2} < \frac{\lambda((a'\sigma(b,\mu(b))^l)^{1/d})}{x} < N_2$. By Lemma 2.16, there is $j \in \mathbb{Z}$ such that $x = \varepsilon^j\lambda((a'\sigma(b,\mu(b))^l)^{1/d})$. Since $\lambda(\varepsilon^j) = \varepsilon^j$, by Lemma 2.15, we have $x \in \lambda((K'(a)^{\mathrm{rc}})^{>0})$. Therefore, in this case, we have $\lambda((K')^{>0})\langle\lambda(\sigma(b,\mu(b)))\rangle_A = \lambda((K'(b,\mu(b))^{\mathrm{rc}})^{>0})$.

By Lemma 2.15, $\lambda(x) \in K'(b,\mu(b))^{\mathrm{rc}}$, so $x \in K'(b,\mu(b))^{\mathrm{rc}}$.

Case 3: $D = 1$. Then one of d_1, d_2 must be 0 and the other must be 1. First suppose $d_1 = 0$ and $d_2 = 1$. Since $d_2 = 1$, there is $y \in K'(b)^{\mathrm{rc}}$ such that $v(y) \notin X$. Since we assume that $d_1 = 0$, $V = W$. Thus, by a similar proof as in Case 2 above, we have $V = X \oplus \mathbb{Q}v(y)$. Let $y = \sigma(b)$, where σ is an $\mathcal{L}_{or}(K')$-definable function. Letting $\tau(x,w) := \sigma(x)$, we also have $y = \tau(b,\mu(b))$. A similar proof as in Case 2 above shows that $\lambda(K'(b,\mu(b))^{\mathrm{rc}}) = A'\langle\tau(b,\mu(b))^{\mathrm{rc}}\rangle_A$.

Next suppose that $d_1 = 1$ and $d_2 = 0$. In this case, we have $z \in K'(b,\mu(b))^{\mathrm{rc}}$ with $v(z) \notin W$. Since $d_2 = 0$, we have $W = X$. By a similar proof as in Case 2 above, $V = W \oplus \mathbb{Q}v(z) = X \oplus \mathbb{Q}v(z)$. Let σ be an $\mathcal{L}_{or}(K')$-definable function such that $z = \sigma(b,\mu(b))$. Again, a similar proof as in Case 2 above shows that $\lambda(K'(b,\mu(b))^{\mathrm{rc}}) = A'\langle\sigma(b,\mu(b))^{\mathrm{rc}}\rangle_A$.

Lastly, we prove that if $b \in A$, we can take $\tau = \mathrm{id}$. We have two cases: $v(b) \in v(K')$ and $v(b) \notin v(K')$. Since we assume $b \in A$, $\mu(b) = 1$. Therefore, if $b \in A$, we have $K'(b,\mu(b))^{\mathrm{rc}} = K'(b)^{\mathrm{rc}}$ and $V = W$.

If $v(b) \in v(K')$, there is $k \in K'$ such that $v(k) = v(b)$. We may assume that $k > 0$ in K'. Thus, there is $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{b}{\lambda(k)} < n$. Therefore, $\frac{b}{\lambda(k)}$ is a finite element of A, so there is $l \in \mathbb{Z}$ such that $b = \varepsilon^l\lambda(k)$. Since K' is closed under λ, $b \in A'$. Now note that since K' is real closed and we assume that $A' = K' \cap A$, A' is pure in A. Therefore, $A'\langle b\rangle_A = A'$. We also have $K'(b)^{\mathrm{rc}} = K'$. Therefore, $K'(b,\mu(b))^{\mathrm{rc}} = K'(b)^{\mathrm{rc}} = K'$. So $\lambda(K'(b,\mu(b))^{\mathrm{rc}}) = \lambda(K') = A'$.

In the second case, we have $V = X \oplus \mathbb{Q}v(b)$ by a similar proof as in Case 3 above. Since $d_1 = 0$ and $d_2 = 1$ in this case, a similar proof as in Case 3 shows that $\lambda(K'(b, \mu(b))^{\mathrm{rc}}) = A'\langle b\rangle_A$. $\square$

In Section 5.1.2, we will take $\mu : K \to K(i)$ to be the function which takes every element of K to 1 when applying this lemma.

Chapter 3

Model-theoretic definitions and lemmas

In this section, we will collect some useful theorems from model theory that we will use in this thesis.

3.1 Real closures

For many of the theorems we prove, it is helpful to be able to generate a real closed field from a subset of a larger real closed field. Moreover, we would like to have a concrete way to describe the real closed field generated in this way. The main theorem in this subsection gives us such a description.

We first recall two characterizations of real closed fields from Marker [21].

Theorem 3.1 (Theorem 3.3.9, [21]). *An ordered field F is real closed if and only if whenever $p(x) \in F[x]$, $a, b \in F$ with $a < b$, and $p(a)p(b) < 0$, there is $c \in F$ such that $a < c < b$ and $p(c) = 0$.*

Theorem 3.2 (Theorem 3.3.5, [21]). *Let F be an ordered field, and let i be an element in a field extension of F satisfying $i^2 = -1$. Then F is real closed if and only if $F(i)$ is algebraically closed.*

From Theorem 3.2, if F is a real closed field and $p \in F[x]$, then there are $k, m \in \mathbb{N}$ and C, $a_1, \ldots, a_m$, $c_1, \ldots, c_k$,$d_1, \ldots, d_k \in F$ such that

$$p(x) = C \prod_{i=1}^{m} (x - a_i) \prod_{j=1}^{k} (x^2 + c_j x + d_j)$$

and for each $j \in \{1, \ldots, k\}$, $x^2 + c_j x + d_j$ is irreducible in $F[x]$.

We are now ready to prove the main theorem in this subsection.

Theorem 3.3. *Let R be a real closed field and let K be a subset of R. Then $K^{rc} = dcl^R(K)$.*

Proof. We first prove that $\mathrm{dcl}^R(K)$ is a real closed field such that $K \subseteq \mathrm{dcl}^R(K)$. This will show that $K^{\mathrm{rc}} \subseteq \mathrm{dcl}^R(K)$.

To prove that $\mathrm{dcl}^R(K)$ is a real closed field, we show that $\mathrm{dcl}^R(K)$ is an ordered field that satisfies the condition in Theorem 3.1. First note that $\mathrm{dcl}^R(K)$ is a subfield of R by definition of $\mathrm{dcl}^R(K)$. Thus, we can make $\mathrm{dcl}^R(K)$ into an ordered field by taking the ordering on $\mathrm{dcl}^R(K)$ to be the restriction of the ordering on R.

Fix $a, b \in \mathrm{dcl}^R(K)$ and a polynomial $p \in \mathrm{dcl}^R(K)[x]$ such that $a < b$ and $p(a)p(b) < 0$. Since R is real closed, there is $c \in R$ such that $a < c < b$ and $p(c) = 0$. Since $p(a)p(b) < 0$, p cannot be a constant polynomial. Let n be the degree of p, so that p has at most n roots in R. We define a function f by setting

$$f(y_0, \ldots, y_n) = \begin{cases} \min\{x : y_n x^n + \ldots + y_1 x + y_0 = 0\}, & y_1 \neq 0 \vee \ldots \vee y_n \neq 0 \\ y_0 & \text{otherwise} \end{cases}$$

Clearly, f is $\mathcal{L}_{or}$-definable. Let $a_0, \ldots, a_n \in \mathrm{dcl}^R(K)$ be such that $p(x) = a_n x^n + \ldots + a_1 x + a_0$. Then there is $c_0 \in R$ such that $c_0 = f(a_0, \ldots, a_n)$ and $a < c_0 < b$. By definition of definable closure, $c_0 \in \mathrm{dcl}^R(K)$. Therefore, $\mathrm{dcl}^R(K)$ is real closed.

Next we show that $\mathrm{dcl}^R(K) \subseteq K^{\mathrm{rc}}$. Let $b \in \mathrm{dcl}^R(K)$. Then there is an $\mathcal{L}_{or}$-formula $\phi(\overline{w}, v)$ and tuple $\overline{c} \in K^{|\overline{w}|}$ such that $R \models \phi(\overline{c}, b)$ and $R \models \forall v(\phi(\overline{c}, v) \to v = b)$. Since RCF has quantifier elimination, there are $\mathcal{L}_{or}$-formulas $\phi_1, \ldots, \phi_n$ such that

$$\mathrm{RCF} \models \phi(\overline{w}, v) \leftrightarrow (\phi_1(\overline{w}, v) \vee \ldots \vee \phi_n(\overline{w}, v))$$

and each $\phi_i(\overline{w}, v)$ is a conjunction of formulas of the form $p(\overline{w}, v) = 0$, $p(\overline{w}, v) < 0$, or $p(\overline{w}, v) > 0$, where $p \in \mathbb{Z}[\overline{w}, v]$. Since $R \models \phi(\overline{c}, b)$, there is $i \in \{1, \ldots, n\}$ such that $R \models \phi_i(\overline{c}, b)$. Since b is the only element of R that satisfies $\phi(\overline{c}, v)$, there must be a polynomial $p \in \mathbb{Z}[\overline{w}, v]$ such that $\phi_i(\overline{c}, v)$ has $p(\overline{c}, v) = 0$ as a conjunct. By Theorem 3.2, all roots of $p(\overline{c}, v)$ lie in $K^{\mathrm{rc}}(i)[v]$. Since R is real closed, R cannot contain i. Therefore, $b \in K^{\mathrm{rc}}$. □

If R is a real closed field and $S \subseteq R(i)$, we will write S^{rc} to denote $(\mathrm{Re}(S))^{\mathrm{rc}}$.

3.2 O-minimality

In this section, we collect some important theorems about o-minimal structures which we will use in Chapter 5 and Chapter 6 in studying the open cores of structures.

We begin by stating the definition of o-minimality.

Definition 3.4. An ordered structure $(R, <, \ldots)$ is *o-minimal* if every definable $X \subseteq R$ is a finite union of intervals with endpoints in $R \cup \{\pm\infty\}$ and points in R.

In the rest of this subsection, let $\mathcal{R}$ be an o-minimal structure.

Throughout this thesis, for an ordered structure $(M, <, \ldots)$, we consider M as a topological space by equipping it with the order topology. For $n > 1$, we consider M^n as a topological space by equipping it with the product topology.

Since $\mathcal{R}$ is o-minimal, the definable subsets of R are finite unions of intervals and points. However, we can also describe the definable subsets of R^m, where $m > 1$. To state the theorem which gives us this description, we first need the following definitions, which are taken from van den Dries [12].

Let $X \subseteq R^m$ be definable. Set

$$C(X) := \{f : X \to R : f \text{ is definable and continuous}\}$$

and

$$C_\infty(X) = C(X) \cup \{-\infty, +\infty\}$$

where $-\infty$ and $+\infty$ are considered to be constant functions on X.

Definition 3.5. Let $(i_1, \ldots, i_m)$ be a sequence of zeros and ones of length m. An $(i_1, \ldots, i_m)$-cell is a definable subset of R^m obtained by induction on m as follows.

1. (a) A (0)-cell is a one-element set $\{r\} \subseteq R$.

   (b) A (1)-cell is an interval $(a, b) \subseteq R$.

2. Suppose we have defined $(i_1, \ldots, i_m)$-cells.

   (a) An $(i_1, \ldots, i_m, 0)$-cell is the graph $\Gamma(f)$ of a function $f \in C(X)$, where X is an $(i_1, \ldots, i_m)$-cell.

   (b) An $(i_1, \ldots, i_m, 1)$-cell is a set of the form

   $$\{(x, r) \in X \times R : f(x) < r < g(x)\}$$

   where X is an $(i_1, \ldots, i_m)$-cell and $f, g \in C_\infty(X)$ with $f(x) < g(x)$ for all $x \in X$.

Definition 3.6. A cell in R^m is an $(i_1, \ldots, i_m)$-cell for some (unique) sequence $(i_1, \ldots, i_m)$. An open cell is a cell which as a set is open in the topology on R^m.

As stated in (2.4) in [12], the open cells are exactly the $(1, 1, \ldots, 1)$-cells.

Definition 3.7. A decomposition of R^m is defined by induction on m as follows.

1. A decomposition of $R^1 = R$ is a collection

$$\{(-\infty, a_1), (a_1, a_2), \ldots, (a_k, +\infty), \{a_1\}, \ldots, \{a_k\}\}$$

where $a_1 < \ldots < a_k$ are points in R.

2. A decomposition of R^{m+1} is a finite partition of R^{m+1} into cells A such that the set of projections $\pi(A)$ is a decomposition of R^m. (Here $\pi : R^{m+1} \to R^m$ is the usual projection map onto the first m coordinates.)

A decomposition $\mathcal{D}$ is said to partition a set $X \subseteq R^m$ if X is a union of cells in $\mathcal{D}$.

The next theorem gives us a way to describe the definable subsets of R^m for $m > 1$.

Theorem 3.8 (Cell decomposition). *Let $X \subseteq R^m$ be a definable set. There is a decomposition of R^m which partitions X.*

Fix a subset $S \subseteq R$. It can be shown that if $X \subseteq R^m$ is definable with parameters from S, then for any cell C in a decomposition of R^m such that $C \subseteq X$, C is also definable with parameters from S.

We also need the following fact about definable closure in o-minimal structures from Pillay and Steinhorn [23]. Let $\mathcal{L}$ be a language expanding $\mathcal{L}_{or}$ and let $\mathcal{M}$ be an $\mathcal{L}$-structure.

Theorem 3.9 ([23]). *The closure operator $dcl^{\mathcal{M}}_{\mathcal{L}_{or}}$ has the exchange property. That is, for $a, b \in M$ and $S \subseteq M$, if $a \in dcl^{\mathcal{M}}_{\mathcal{L}_{or}}(S \cup \{b\})$ and $a \notin dcl^{\mathcal{M}}_{\mathcal{L}_{or}}(S)$, then $b \in dcl^{\mathcal{M}}_{\mathcal{L}_{or}}(S \cup \{a\})$.*

3.3 Orientation and Mann axioms

We now define two important sets of axioms: orientation axioms and Mann axioms. We will make use of these axioms in our axiomatizations in Chapter 5 and Chapter 6. In real closed fields, the type of an element is determined by its cut over $\mathbb{Q}$. Intuitively, the orientation axioms should determine ordered ring types. The Mann axioms give us a first-order way to specify the solutions of certain linear equations in groups with the Mann property.

Let R be a real closed field and let Γ and Δ be subgroups of $\mathbb{S}(R)$ and $R^{>0}$ respectively of finite rank. We identify Δ with the subgroup $\{(\delta, 0) : \delta \in \Delta\}$ of R^2. Let $\Gamma\Delta = \{\gamma\delta : \gamma \in \Gamma, \delta \in \Delta\}$. Since both Γ and Δ are assumed to have finite rank, it is not hard to show that $\Gamma\Delta$ also has finite rank. By Section 2.1.1, $\Gamma\Delta$ has the Mann property.

Given any polynomial $Q(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$ and tuple $\gamma\delta := (\gamma_1\delta_1, \ldots, \gamma_n\delta_n)$ of elements of $\Gamma\Delta$, we say the *ordering axiom for* $\gamma\delta$ *and* Q is the sentence

$$Q(\mathrm{Re}(\gamma_1\delta_1), \ldots, \mathrm{Re}(\gamma_n\delta_n)) > 0$$

if this holds in $\mathbb{R}$, and otherwise it is the sentence

$$Q(\mathrm{Re}(\gamma_1\delta_1), \ldots, \mathrm{Re}(\gamma_n\delta_n)) \leq 0.$$

The *orientation axioms of* $\Gamma\Delta$ consist of the collection of ordering axioms for each n, each polynomial $Q \in \mathbb{Z}[x_1, \ldots, x_n]$, and each tuple $\gamma\delta \in (\Gamma\Delta)^n$.

For every linear equation

$$a_1x_1 + \ldots + a_nx_n = 1, (n \geq 1, a_1, \ldots, a_n \in \mathbb{Q}^\times)$$

take the finite list of its nondegenerate solutions in $\Gamma\Delta$, say

$$\gamma_1\delta_1 = (\gamma_{11}\delta_{11}, \ldots, \gamma_{1n}\delta_{1n}), \ldots, \gamma_k\delta_k = (\gamma_{k1}\delta_{k1}, \ldots, \gamma_{kn}\delta_{kn}).$$

For n-tuples $y = (y_1, \ldots, y_n)$ and $z = (z_1, \ldots, z_n)$ and $\delta \in \Delta$ let $P(\delta^{-1}y, \delta^{-1}z)$ abbreviate

$$P(\delta^{-1}y_1, \delta^{-1}z_1) \wedge \ldots \wedge P(\delta^{-1}y_n, \delta^{-1}z_n)$$

and let $(y, z) = \gamma_j\delta_j$ abbreviate

$$\begin{aligned} y_1 = \mathrm{Re}(\gamma_{j1}\delta_{j1}) \wedge \ldots \wedge y_n = \mathrm{Re}(\gamma_{jn}\delta_{jn}) \\ \wedge z_1 = \mathrm{Im}(\gamma_{j1}\delta_{j1}) \wedge \ldots \wedge z_n = \mathrm{Im}(\gamma_{jn}\delta_{jn}). \end{aligned}$$

Let *the Mann axiom of* $\Gamma\Delta$ *corresponding to the equation* $a_1x_1 + \ldots + a_nx_n = 1$ be the sentence

$$\forall y \forall z \left[\forall b \left(V(b) \wedge P(by, bz) \wedge \sum_{i=1}^{n} a_i y_i = 1 \wedge \sum_{i=1}^{n} a_i z_i = 0 \wedge \right.\right.$$
$$\left.\left. \bigwedge_I (\sum_{i \in I} a_i y_i \neq 0 \vee \sum_{i \in I} a_i z_i \neq 0) \right) \to \bigvee_{j=1}^{k} (y, z) = \gamma_j\delta_j \right].$$

Here the conjunction $\bigwedge_I$ is taken over all nonempty proper subsets I of $\{1, \ldots, n\}$.

Let K be a real closed field. Suppose the predicate P is interpreted as a subgroup $G \subseteq \mathbb{S}^1(K)$ and the predicate V is interpreted as a subgroup $A \subseteq K^{>0}$. In this setting, the Mann axiom of $\Gamma\Delta$ corresponding to the equation

$$a_1 x_1 + \ldots + a_n x_n = 1$$

can be interpreted as follows. Let $\gamma_1\delta_1, \ldots, \gamma_n\delta_n$ be the solutions to this equation in $\Gamma\Delta$. Suppose we have $(y_1, z_1), \ldots, (y_n, z_n) \in GA$ such that

$$a_1(y_1 + iz_1) + \ldots + a_n(y_n + iz_n) = 1.$$

In particular, we must have $\sum_{i=1}^n a_i z_i = 0$ and $\sum_{i=1}^n a_i y_i = 1$. Suppose also that for all proper subsets $I \subsetneq \{1, \ldots, n\}$, we have $\sum_{i \in I} a_i y_i \neq 0$ and $\sum_{i \in I} a_i z_i \neq 0$. That is, $((y_1, z_1), \ldots, (y_n, z_n))$ is a nondegenerate solution of the equation $a_1 x_1 + \ldots + a_n x_n = 1$. Then letting $y = (y_1, \ldots, y_n)$ and $z = (z_1, \ldots, z_n)$, we must have $(y, z) = \gamma_j \delta_j$ in the sense defined above.

Let K, L be real closed fields, and let C be a set of constants not in $\mathcal{L}_{or}$. Let $(K, (c)_{c \in C})$ denote the structure K together with interpretations of all the constants in C, and define $(L, (c)_{c \in C})$ similarly.

From quantifier elimination for real closed fields, we obtain the following lemma which we will use in Chapter 5 and Chapter 6.

Lemma 3.10. *Let K, L be real closed fields, and let C be a set of constants not in $\mathcal{L}_{or}$. Suppose that for all $p \in \mathbb{Z}[x_1, \ldots, x_n]$ and all $c_1, \ldots, c_n \in C$, $K \models p(c_1, \ldots, c_n) > 0$ if and only if $L \models p(c_1, \ldots, c_n) > 0$. Then $(K, (c)_{c \in C}) \equiv (L, (c)_{c \in C})$.*

3.4 Quantifier elimination lemmas

In this subsection, we prove some further quantifier elimination lemmas necessary for the proofs of Theorem 5.7 and Theorem 6.17. The main tool for the first result is Lemma 2.23 (see also Section 8 of [17]).

In the following two lemmas, let $\mathcal{L}_1$ be the language of oriented monoids together with constants from an infinite multiplicative subgroup $\Gamma \subseteq \mathbb{S}^1$. That is, $\mathcal{L}_1 = \{\mathcal{O}, \cdot, 1, (\gamma)_{\gamma \in \Gamma}\}$, where $\mathcal{O}$ is a ternary predicate. Fix a function e from the set of prime numbers to $\mathbb{N}$. Let $\mathcal{L}_2 = \mathcal{L}_1 \cup \{E_n : n > 0\}$, where each E_n is a unary predicate with defining axiom

$$\sigma_n := \forall z (E_n(z) \leftrightarrow \exists y (z = y^n)).$$

Lemma 3.11. *Let $\Sigma_1(e)$ be the theory of regularly dense oriented abelian groups G containing Γ as a subgroup such that $[p]G = p^{e(p)}$ for all primes p and $G_{tor} = \Gamma_{tor}$. Let $\Sigma_2 = \{\sigma_n : n > 0\}$. Then the $\mathcal{L}_2$-theory $\Sigma(e) := \Sigma_1(e) \cup \Sigma_2$ admits quantifier elimination.*

Proof. Let G, H be $\mathcal{L}_2$-structures such that $G, H \models \Sigma(e)$ and H is κ^+-saturated for some $\kappa \geq |G|$. Let G' be a proper $\mathcal{L}_2$-substructure of G and let $f : G' \to H$ be an embedding. Let $g \in G \setminus G'$, and let H' be the substructure of H with $H' = f(G')$. We will find $h \in H \setminus H'$ such that f extends to an $\mathcal{L}_2$-isomorphism $f' : G'\langle g\rangle_G \to H'\langle h\rangle_H$.

Note that by definition of $\Sigma(e)$, we have $[p]G = [p]H$ for all primes p. Since $G' \subseteq G$, we also have $\Gamma_{\text{tor}} \subseteq G'_{\text{tor}} \subseteq G_{\text{tor}} = \Gamma_{\text{tor}}$. Therefore, $G'_{\text{tor}} = G_{\text{tor}}$. Similarly, $H'_{\text{tor}} = H_{\text{tor}}$. If $g' \in G'$ has an nth root in G, then $G \models E_n(g')$. Since $G' \subseteq G$, we also have $G' \models E_n(g')$, so G' is pure in G. Likewise, H' is pure in H.

By Lemma 2.23, there is $h \in H$ such that there is an oriented group isomorphism $f' : G'\langle g\rangle_G \to H'\langle h\rangle_H$ extending f and taking g to h. Note that since f' extends f, we have $f'(\gamma) = \gamma$ for all $\gamma \in \Gamma$. Therefore, f' is an $\mathcal{L}_1$-isomorphism between $G'\langle g\rangle_G$ and $H'\langle h\rangle_H$. We now check that f' is an $\mathcal{L}_2$-isomorphism.

Suppose $G'\langle g\rangle_G \models E_n(b)$ for some $b \in G'\langle g\rangle_G$. By definition of E_n, $b = y^n$ for some $y \in G'\langle g\rangle_G$. Since f' is an oriented group isomorphism, we have $f'(b) = (f'(y))^n$. So $H'\langle h\rangle_H \models E_n(f'(b))$. Conversely, if $H'\langle h\rangle_H \models E_n(f'(b))$ for some $b \in G'\langle g\rangle_G$, then $G'\langle g\rangle_G \models E_n(b)$. Therefore, f' is an $\mathcal{L}_2$-isomorphism between $G'\langle g\rangle_G$ and $H'\langle h\rangle_H$.

Since we have found an $\mathcal{L}_2$-isomorphism f' properly extending f, Σ has quantifier elimination. □

Lemma 3.12. *Let Φ be a set of $\mathcal{L}_2$-sentences that axiomatizes the class of abelian groups. Every atomic $\mathcal{L}_2$-formula $\varphi(x_1, \dots, x_n)$ is equivalent in Φ to a formula with one of the following forms:*

$$\gamma^k x_1^{k_1} \dots x_n^{k_n} = 1$$

$$\mathcal{O}(\gamma_1^k x_1^{k_1} \dots x_n^{k_n}, \gamma_2^l y_1^{l_1} \dots y_m^{l_m}, \gamma_3^p z_1^{i_1} \dots z_p^{i_p})$$

$$E_d(\gamma^k x_1^{k_1} \dots x_n^{k_n})$$

where $k_1, \dots, k_n$, $l_1, \dots, l_m$, $i_1, \dots, i_p \in \mathbb{Z}$, k, l, i are tuples of elements of $\mathbb{Z}$, d is a positive integer, and $\gamma, \gamma_1, \gamma_2, \gamma_3$ are tuples of elements from Γ.

Proof. Since any $\mathcal{L}_2$-structure which models Φ is an abelian group, one can show by induction on terms that in any model of Φ, every $\mathcal{L}_2$-term is equal to a term of the form $\gamma^k x_1^{k_1} \dots x_n^{k_n}$. Thus, it is clear that every $\mathcal{L}_2$-atomic formula must be equivalent in Φ to a formula with one of the above forms. □

We next recall some results for regularly discrete abelian groups. In the following two lemmas, let $\mathcal{L}_3 = \{\cdot, <, 1, \varepsilon\} \cup \{D_n : n > 0\}$, where each D_n is a unary predicate. For $n > 0$, let

$$\tau_n := \forall x(D_n(x) \leftrightarrow \exists y(x = y^n)).$$

Lemma 3.13. *Let T_3 be the $\mathcal{L}_3$-theory of regularly discrete abelian groups A with ε the smallest element larger than 1, together with the set of sentences $\{\tau_n : n > 0\}$. Then T_3 admits quantifier elimination.*

Proof. By Lemma 2.10, for each $n \geq 1$ and each $a \in A$, there is $i \in \{1, \ldots, n\}$ such that $a\varepsilon^i \in A^{[n]}$. Therefore, the theory T_3 includes the sentence

$$\forall a(D_n(a\varepsilon) \vee \ldots \vee D_n(a\varepsilon^n))$$

for each $n \geq 1$. Since the theory of $\mathbb{Z}$-groups admits quantifier elimination, T_3 admits quantifier elimination. □

From this, the following lemma follows easily by a similar proof as Lemma 3.12.

Lemma 3.14. *Let Φ' be a set of $\mathcal{L}_3$-sentences that axiomatizes the class of abelian groups. Every $\mathcal{L}_3$-atomic formula $\varphi(x_1, \ldots, x_n)$ is equivalent in Φ' to a formula with one of the following forms:*

$$\delta^k x_1^{k_1} \ldots x_n^{k_n} = 1$$
$$\delta^k x_1^{k_1} \ldots x_n^{k_n} < 1$$
$$D_d(\delta^k x_1^{k_1} \ldots x_n^{k_n})$$

where $k_1, \ldots, k_n \in \mathbb{Z}$, k is a tuple of elements of $\mathbb{Z}$, $d > 0$, and δ is a tuple of elements from $\varepsilon^{\mathbb{Z}}$.

We will need the following lemma for the proof of Theorem 6.17.

Fix a function f from the set of prime numbers to $\mathbb{N}$. Fix an infinite multiplicative subgroup $\Xi \subseteq \mathbb{R}^{>0}$. Let $\mathcal{L}_1 = \mathcal{L}_{\text{oab}} \cup \{\xi\}_{\xi \in \Xi} \cup \{D_n\}_{n \in \mathbb{N}}$, where D_n is a unary predicate for each n. Let $\Sigma_1(f)$ be the theory of regularly dense ordered abelian groups B containing Ξ as a subgroup such that $[p]B = p^{f(p)}$ for all primes p. Let Σ_2 consist of the set of sentences

$$\forall z(D_n(z) \leftrightarrow \exists y(z = y^n)).$$

Lemma 3.15. *The $\mathcal{L}_1$-theory $\Sigma_1(f) \cup \Sigma_2$ admits quantifier elimination.*

Proof. This follows from the remarks before the proof of Theorem 7.1 in [13]. □

Chapter 4

Density of subgroups of $\mathbb{C}^\times$

In this section, we will prove several results about the density of groups in $\mathbb{C}$ which we will use in Chapter 6.

For vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, let $\mathbf{x} \cdot \mathbf{y}$ denote the dot product of $\mathbf{x}$ and $\mathbf{y}$. Let $\|\mathbf{x}\|_2$ denote the Euclidean norm of $\mathbf{x}$.

Throughout this section, we will use the following version of Kronecker's theorem. This version follows directly from Theorem 24 in Siegel [25].

Theorem 4.1 (Kronecker's theorem)**.** *Let A be an $m \times n$ matrix with real entries and let $\boldsymbol{b} \in \mathbb{R}^m$. The following are equivalent:*

1. *For every $\boldsymbol{y} \in \mathbb{R}^m$ such that $A^T \boldsymbol{y} \in \mathbb{Z}^n$, we have $\boldsymbol{b} \cdot \boldsymbol{y} \in \mathbb{Z}$;*
2. *For all $\varepsilon > 0$, there is $\boldsymbol{g} \in \mathbb{Z}^n$ such that $\|A\boldsymbol{g} - \mathbf{b}\|_2 \leq \varepsilon$.*

4.1 Finitely generated subgroups of $\mathbb{C}^\times$

In this subsection, we prove some results relating the density of finitely generated subgroups of $\mathbb{C}$ to algebraic relations on the generators.

We start by considering a subgroup generated by two elements, as we wanted to study expansions of $\overline{\mathbb{R}}$ by subgroups of $\mathbb{C}$ generated by two elements. However, the proofs of the main theorems in this subsection also go through for finitely generated groups.

Let $G := (ae^{i\varphi})^{\mathbb{Z}}(be^{i\psi})^{\mathbb{Z}}$, where $a, b \in \mathbb{R}^{>0}$ and $\varphi, \psi \in \mathbb{R}$. Let A be the 2×3 matrix

$$A = \begin{bmatrix} \ln(a) & \ln(b) & 0 \\ \varphi & \psi & 2\pi \end{bmatrix}.$$

Lemma 4.2. *The group G is dense in $\mathbb{C}$ if and only if for all $\boldsymbol{v} \in \mathbb{R}^2$, if $\boldsymbol{y} \in \mathbb{R}^2$ satisfies $A^T \boldsymbol{y} \in \mathbb{Z}^3$, then $\boldsymbol{v} \cdot \boldsymbol{y} \in \mathbb{Z}$.*

Proof. Suppose that for all $\mathbf{v} \in \mathbb{R}^2$, if $\mathbf{y} \in \mathbb{R}^2$ satisfies $A^T\mathbf{y} \in \mathbb{Z}^3$, then $\mathbf{v} \cdot \mathbf{y} \in \mathbb{Z}$. Fix $\varepsilon > 0$ and let $c = r_0 e^{i\tau_0}$ be an arbitrary complex number, where $r_0 \in \mathbb{R}^{\geq 0}$ and $\tau_0 \in \mathbb{R}$. We may assume that $r_0 \neq 0$. We want to find $n_1, n_2 \in \mathbb{Z}$ such that $\left|(ae^{i\varphi})^{n_1}(be^{i\psi})^{n_2} - c\right| < \varepsilon$. For all $k_1, k_2 \in \mathbb{Z}$, we have

$$\begin{aligned}\left|(ae^{i\varphi})^{k_1}(be^{i\psi})^{k_2} - c\right| < \varepsilon &\Leftrightarrow \left|e^{k_1(\ln(a)+i\varphi)+k_2(\ln(b)+i\psi)} - e^{\ln(r_0)}e^{i\tau_0}\right| < \varepsilon \\ &\Leftrightarrow \left|e^{i\tau_0}\right|\left|e^{k_1(\ln(a)+i\varphi)+k_2(\ln(b)+i\psi)-i\tau_0} - e^{\ln(r_0)}\right| < \varepsilon.\end{aligned}$$

Thus, for all $k_1, k_2 \in \mathbb{Z}$,

$$\left|(ae^{i\varphi})^{k_1}(be^{i\psi})^{k_2} - c\right| < \varepsilon \Leftrightarrow \left|e^{k_1(\ln(a)+i\varphi)+k_2(\ln(b)+i\psi)-i\tau_0-\ln(r_0)} - 1\right| < \frac{\varepsilon}{r_0}. \tag{4.1}$$

Since $\lim_{z\to 0} e^z = 1$, let $\delta > 0$ be such that for all $z \in \mathbb{C}$, if $|z| < \delta$, then $|e^z - 1| < \frac{\varepsilon}{r_0}$. To find $n_1, n_2 \in \mathbb{Z}$ such that the right side of Eq. (4.1) holds, it suffices to find $n_1, n_2, n_3 \in \mathbb{Z}$ such that

$$|n_1 \ln(a) + n_2 \ln(b) - \ln(r_0)| < \frac{\delta}{2} \text{ and } |n_1\varphi + n_2\psi + n_3 2\pi - \tau_0| < \frac{\delta}{2}.$$

For if we can find such n_1, n_2, n_3, then

$$|(n_1 \ln(a) + n_2 \ln(b) - \ln(r_0)) + i(n_1\varphi + n_2\psi + n_3 2\pi - \tau_0)| < \delta.$$

Since $e^{2\pi n_3 i} = 1$, letting $z = n_1(\ln(a) + i\varphi) + n_2(\ln(b) + i\psi) - i\tau_0 - \ln(r_0)$, our choice of δ gives us that $|e^z - 1| < \frac{\varepsilon}{r_0}$. Hence $\left|(ae^{i\varphi})^{n_1}(be^{i\psi})^{n_2} - c\right| < \varepsilon$, as desired.

Let

$$\mathbf{b} = \begin{bmatrix} \ln(r_0) \\ \tau_0 \end{bmatrix}.$$

By Kronecker's theorem, there are $n_1, n_2, n_3 \in \mathbb{Z}$ such that

$$\left((n_1 \ln(a) + n_2 \ln(b) - \ln(r_0))^2 + (n_1\varphi + n_2\varphi + n_3 2\pi - \tau_0)^2\right)^{1/2} < \frac{\delta}{2}.$$

Therefore, we have

$$|n_1 \ln(a) + n_2 \ln(b) - \ln(r_0)| < \frac{\delta}{2} \text{and} \, |n_1\varphi + n_2\psi + n_3 2\pi - \tau_0| < \frac{\delta}{2}.$$

Conversely, suppose there are $\mathbf{v}, \mathbf{y} \in \mathbb{R}^2$ such that $A^T\mathbf{y} \in \mathbb{Z}^3$ but $\mathbf{v} \cdot \mathbf{y} \notin \mathbb{Z}$. Let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and let $r_0 > 0$ be such that $v_1 = \ln(r_0)$. By Kronecker's theorem, there is $\varepsilon > 0$ such that for all $\mathbf{z} \in \mathbb{Z}^3$, $\|A\mathbf{z} - \mathbf{v}\|_2 > \varepsilon$. That is, for all $n_1, n_2, n_3 \in \mathbb{Z}$,

$$((n_1 \ln(a) + n_2 \ln(b) - \ln(r_0))^2 + (n_1\varphi + n_2\psi + n_3 2\pi - v_2)^2)^{1/2} > \varepsilon.$$

Let

$$S = \{n_1 \ln(a) + n_2 \ln(b) - \ln(r_0) + i(n_1\varphi + n_2\psi - v_2) : n_1, n_2 \in \mathbb{Z}\}.$$

Kronecker's theorem gives us that for all $z \in S$ and all $n \in \mathbb{Z}$,

$$|\mathrm{Re}(z)| + |\mathrm{Im}(z) - (2\pi n)| \geq |z - i(2\pi n)| > \varepsilon.$$

For any $z \in S$ such that $|\mathrm{Re}(z)| > \frac{\varepsilon}{2}$, there is $\alpha > 0$ such that $|e^z - 1| > \frac{\alpha}{r_0}$. (The map $w \mapsto e^w$ takes the strip $\{z \in \mathbb{C} : |\mathrm{Re}(z)| \leq \varepsilon/2\}$ to the annulus with inner radius $e^{-\varepsilon/2}$ and outer radius $e^{\varepsilon/2}$.) By Eq. (4.1), we have $\left|(ae^{i\varphi})^{k_1}(be^{i\psi})^{k_2} - r_0e^{iv_2}\right| > \alpha$ for any $k_1, k_2 \in \mathbb{Z}$ such that $|k_1 \ln(a) + k_2 \ln(b) - \ln(r_0)| > \frac{\varepsilon}{2}$.

For any $z \in S$ with $|\mathrm{Re}(z)| \leq \varepsilon/2$, we must have $|\mathrm{Im}(z) - 2\pi n| > \varepsilon/2$ for all $n \in \mathbb{Z}$. Therefore, we can find $\beta > 0$ such that for any $z \in S$ with $|\mathrm{Re}(z)| \leq \varepsilon/2$, $|e^z - 1| > \beta/r_0$. Thus, there is $\delta > 0$ such that for all $z \in S$, $|e^z - 1| > \delta/r_0$. By Eq. (4.1) (with $c = r_0e^{iv_2}$), for all $g \in G$, we have $|g - c| > \delta$.

Therefore, G is not dense in $\mathbb{C}$ in this case. ∎

Lemma 4.3. *If there is $\boldsymbol{y} \in \mathbb{R}^2$ with $\boldsymbol{y} \neq \boldsymbol{0}$ such that $A^T\boldsymbol{y} \in \mathbb{Z}^3$, then there is $\boldsymbol{v} \in \mathbb{R}^2$ such that $\boldsymbol{v} \cdot \boldsymbol{y} \notin \mathbb{Z}$.*

Proof. Let $\mathbf{y} := \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ be an element of $\mathbb{R}^2$ such that $A^T\mathbf{y} \in \mathbb{Z}^3$. Let k_1, k_2, k_3 be elements of $\mathbb{Z}$ such that $A^T\mathbf{y} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$. Then we must have $y_2 = \frac{k_3}{2\pi}$. If $k_3 \neq 0$, then taking $\mathbf{v} := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we get that $\mathbf{v} \cdot \mathbf{y} \notin \mathbb{Z}$. If $k_3 = 0$, then we have $y_2 = 0$, so by assumption $y_1 \neq 0$. If $y_1 \in \mathbb{Q}$, then take $\mathbf{v} := \begin{bmatrix} r \\ 0 \end{bmatrix}$, where $r \in \mathbb{R} \setminus \mathbb{Q}$. If $y_1 \notin \mathbb{Q}$, then take $\mathbf{v} := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. ∎

Corollary 4.4. *The group G is dense in $\mathbb{C}$ if and only if for all $\boldsymbol{y} \in \mathbb{R}^2 \setminus \{0\}$, $A^T\boldsymbol{y} \notin \mathbb{Z}^3$.*

Proof. Suppose G is dense in $\mathbb{C}$ and suppose for a contradiction that there is $\mathbf{y} \in \mathbb{R}^2 \setminus \{0\}$ such that $A^T\mathbf{y} \in \mathbb{Z}^3$. By Lemma 4.3, there is $\mathbf{v} \in \mathbb{R}^2$ such that $\mathbf{v} \cdot \mathbf{y} \notin \mathbb{Z}$. But by Lemma 4.2, if G is dense in $\mathbb{C}$, then we must have $\mathbf{v} \cdot \mathbf{y} \in \mathbb{Z}$.

Now suppose that for all $\mathbf{y} \in \mathbb{R}^2 \setminus \{0\}$ we have $A^T\mathbf{y} \notin \mathbb{Z}^3$. To prove that G is dense in $\mathbb{C}$, it suffices by Lemma 4.2 to show that for any $\mathbf{v} \in \mathbb{R}^2$ and $\mathbf{y} \in \mathbb{R}^2$, $A^T\mathbf{y} \in \mathbb{Z}^3$ implies $\mathbf{v} \cdot \mathbf{y} \in \mathbb{Z}$. Fix $\mathbf{v} \in \mathbb{R}^2$. If $\mathbf{y} = \mathbf{0}$, it is clear that $\mathbf{v} \cdot \mathbf{y} \in \mathbb{Z}$. If $\mathbf{y} \neq \mathbf{0}$, then by assumption $A^T\mathbf{y} \notin \mathbb{Z}^3$. In either case, the implication holds. □

Although we have stated these theorems for a group generated by two complex numbers, the above proofs work for any finitely generated subgroup of $\mathbb{C}$. Let $G' := (a_1 e^{i\varphi_1})^{\mathbb{Z}} \ldots (a_n e^{i\varphi_n})^{\mathbb{Z}}$, where $a_i \in \mathbb{R}^{>0}$ and $\varphi_i \in [0, 2\pi)$ for $i \in \{1, \ldots, n\}$. That is, G' is the multiplicative subgroup of $\mathbb{C}$ generated by $a_1 e^{i\varphi_1}, \ldots, a_n e^{i\varphi_n}$. Let $A(G')$ be the $2 \times (n+1)$-matrix given by

$$A(G') := \begin{bmatrix} \ln(a_1) & \ln(a_2) & \ldots & \ln(a_n) & 0 \\ \varphi_1 & \varphi_2 & \ldots & \varphi_n & 2\pi \end{bmatrix}$$

Using Kronecker's theorem, the proofs of Lemma 4.2, Lemma 4.3, and Corollary 4.4 go through when G and A are replaced by G' and $A(G')$.

The following propositions are well-known, but we state them here for completeness.

Proposition 4.5. *The group $a^{\mathbb{Z}} b^{\mathbb{Z}}$ is dense in $\mathbb{R}^{>0}$ if and only if $\frac{\ln(a)}{\ln(b)} \notin \mathbb{Q}$.*

Proposition 4.6. *The group $(e^{i\varphi})^{\mathbb{Z}}$ is dense in $\mathbb{S}^1$ if and only if $\varphi \notin 2\pi\mathbb{Q}$.*

Proof. If $\varphi \in 2\pi\mathbb{Q}$, then $(e^{i\varphi})^{\mathbb{Z}}$ is finite, so it is clearly not dense in $\mathbb{S}^1$.

Now suppose $\varphi \notin 2\pi\mathbb{Q}$. Then $\frac{\varphi}{2\pi} = r$, where $r \notin \mathbb{Q}$. It can be shown that the set $\{nr \mod 1 : n \in \mathbb{Z}\}$ is dense in $[0, 1)$. Since the function $x \mapsto e^{2\pi i x}$ is continuous and maps $[0, 1)$ onto $\mathbb{S}^1$, $(e^{i\varphi})^{\mathbb{Z}}$ is dense in $\mathbb{S}^1$. □

4.2 Dense graphs

In this subsection, let $H = (ae^{i\varphi})^{\mathbb{Z}} b^{\mathbb{Z}}$, where $a, b \in \mathbb{R}^{\times}$, $a, b > 1$, and $\frac{\ln(a)}{\ln(b)} \notin \mathbb{Q}$. Let $m_H : H \to \mathbb{R}^{>0}$ be the map defined by $m_H(z) = |z|$. Thus, for $(ae^{i\varphi})^n b^m \in H$, $m_H((ae^{i\varphi})^n b^m) = a^n b^m$. Let $\pi_H : H \to \mathbb{S}^1$ be projection onto the unit circle. That is, for $z \in H$, $\pi(z) = z/|z|$.

Definition 4.7. We define a function $\rho : a^{\mathbb{Z}} b^{\mathbb{Z}} \to \mathbb{S}^1$ by $\rho = \pi \circ m_H^{-1}$. For $n, m \geq 1$, $\rho(a^n b^m) = e^{i\varphi n}$. In particular, ρ is definable in $(\overline{\mathbb{R}}, H)$.

It is not hard to check that ρ is a group homomorphism on $a^{\mathbb{Z}} b^{\mathbb{Z}}$ with kernel $b^{\mathbb{Z}}$.

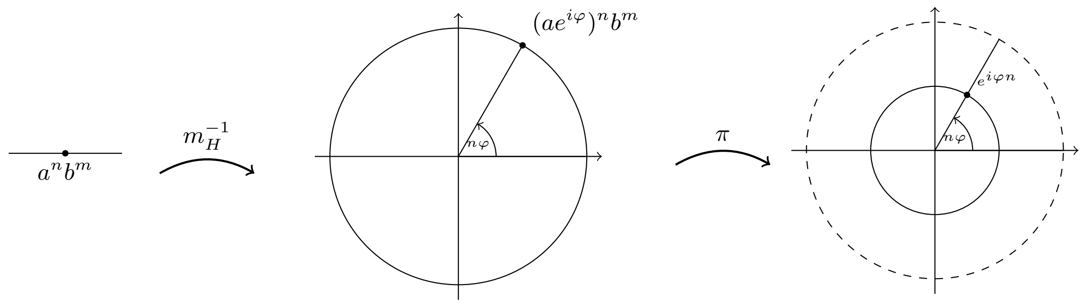

Figure 4.1: A visualization of $\pi \circ m_H^{-1}$ showing that $\rho(a^n b^m) = e^{i\varphi n}$, where $n, m \in \mathbb{Z}$.

The main result of this subsection is the fact that the group H is dense in $\mathbb{C}$ if and only if the graph of ρ is dense in $\mathbb{C}$.

Let

$$B = \begin{bmatrix} \ln(a) & \ln(b) & 0 \\ \varphi & 0 & 2\pi \end{bmatrix}.$$

Proposition 4.8. *The graph of ρ is dense in $\mathbb{R}^{>0} \times \mathbb{S}^1$ if and only if for all $\boldsymbol{v} \in \mathbb{R}^2$, if $\boldsymbol{y} \in \mathbb{R}^2$ satisfies $B^T \boldsymbol{y} \in \mathbb{Z}^3$, then $\boldsymbol{v} \cdot \boldsymbol{y} \in \mathbb{Z}$.*

Proof. Suppose that for all $\mathbf{v} \in \mathbb{R}^2$, if $\mathbf{y} \in \mathbb{R}^2$ satisfies $B^T \mathbf{y} \in \mathbb{Z}^3$, then $\mathbf{v} \cdot \mathbf{y} \in \mathbb{Z}$. Let $c = e^{i\tau}$ be an arbitrary element of $\mathbb{S}^1$ and $r > 0$ an arbitrary positive real number. To show that the graph of ρ is dense in $\mathbb{R}^{>0} \times \mathbb{S}^1$, it suffices to show that for any $\varepsilon > 0$, there are $n_1, n_2 \in \mathbb{Z}$ such that $|a^{n_1} b^{n_2} - r| < \varepsilon$ and $\left|e^{i\varphi n_1} - e^{i\tau}\right| < \varepsilon$. The first condition is equivalent to

$$\left|e^{n_1 \ln(a)} e^{n_2 \ln(b)} e^{-\ln(r)} - 1\right| < \varepsilon / r$$

and the second condition is equivalent to

$$\left|e^{i(n_1 \varphi - \tau)} - 1\right| < \varepsilon.$$

Let $\varepsilon' = \max\{\varepsilon/r, \varepsilon\}$. In the complex numbers, we have $\lim_{z \to 0} e^z = 1$. Thus, there is $\delta > 0$ be such that for all $z \in \mathbb{C}$, if $|z| < \delta$, then $|e^z - 1| < \varepsilon'$. So if we can find $n_1, n_2, n_3 \in \mathbb{Z}$ such that

$$|n_1 \ln(a) + n_2 \ln(b) - \ln(r)| < \delta$$

and

$$|n_1 \varphi + n_3(2\pi) - \tau| < \delta$$

then since $e^{2\pi n_3 i} = 1$, we are done. By assumption, we can apply Kronecker's theorem to find $n_1, n_2, n_3 \in \mathbb{Z}$ satisfying these inequalities.

Conversely, suppose there are $\mathbf{v}, \mathbf{y} \in \mathbb{R}^2$ such that $B^T\mathbf{y} \in \mathbb{Z}^3$ but $\langle \mathbf{v}, \mathbf{y} \rangle \notin \mathbb{Z}$. Let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and let $r > 0$ be such that $v_1 = \ln(r)$. By Kronecker's theorem, there is $\varepsilon > 0$ such that for all $n_1, n_2, n_3 \in \mathbb{Z}$,

$$((n_1 \ln(a) + n_2 \ln(b) - \ln(r))^2 + (n_1\varphi + n_3(2\pi) - v_2)^2)^{1/2} > \varepsilon.$$

Let

$$S = \{(n_1 \ln(a) + n_2 \ln(b) - \ln(r)) + i(n_1\varphi - v_2) : n_1, n_2 \in \mathbb{Z}\}.$$

Since $\left|e^{i(n_1\varphi - v_2)}\right| = 1$, for any $n_1, n_2 \in \mathbb{Z}$, we have

$$\left|e^{n_1 \ln(a)} e^{n_2 \ln(b)} e^{-\ln(r)} - 1\right| = \left|e^{n_1 \ln(a) + n_2 \ln(b) - \ln(r) + i(n_1\varphi - v_2)} - e^{i(n_1\varphi - v_2)}\right|.$$

Therefore,

$$\left|e^{n_1 \ln(a) + n_2 \ln(b) - \ln(r) + i(n_1\varphi - v_2)} - 1\right| \leq$$

$$\left|e^{n_1 \ln(a) + n_2 \ln(b) - \ln(r) + i(n_1\varphi - v_2)} - e^{i(n_1\varphi - v_2)}\right| + \left|e^{i(n_1\varphi - v_2)} - 1\right|. \tag{4.2}$$

By a similar proof as in Lemma 4.2, we can show that there is $\delta > 0$ such that for any $z \in S$, $|e^z - 1| > \delta$. By Eq. (4.2), we must have

$$\left|e^{i(n_1\varphi - v_2)} - 1\right| > \delta/2 \tag{4.3}$$

or

$$\left|e^{n_1 \ln(a) + n_2 \ln(b) - \ln(r) + i(n_1\varphi - v_2)} - e^{i(n_1\varphi - v_2)}\right| > \delta/2 \tag{4.4}$$

for any $n_1, n_2 \in \mathbb{Z}$.

If Eq. (4.3) holds, let V be the open ball of radius $\delta/2$ around e^{iv_2} inside $\mathbb{S}^1$. By Eq. (4.3), for all $n_1 \in \mathbb{Z}$, $e^{in_1\varphi} \notin V$. Let U be any (nonempty) open subset of $\mathbb{R}^{>0}$. Then for all $n_1, n_2 \in \mathbb{Z}$, we have $(a^{n_1}b^{n_2}, e^{i\varphi n_1}) \notin U \times V$.

Similarly, if Eq. (4.4) holds, let U be the open interval of radius $\delta/2$ around r in $\mathbb{R}^{>0}$. Let V be any open subset of $\mathbb{S}^1$. Again, for all $n_1, n_2 \in \mathbb{Z}$, we have $(a^{n_1}b^{n_2}, e^{i\varphi n_1}) \notin U \times V$.

In either case, the graph of ρ is not dense in $\mathbb{R}^{>0} \times \mathbb{S}^1$. □

Thus, by a similar proof as in Corollary 4.4,

Corollary 4.9. *The graph of ρ is dense in $\mathbb{C}$ if and only if for all $\boldsymbol{y} \in \mathbb{R}^2 \setminus \{0\}$, $B^T\boldsymbol{y} \notin \mathbb{Z}^3$.*

This gives us the following corollary.

Corollary 4.10. *The graph of ρ is dense in $\mathbb{C}$ if and only if $\left\{\frac{\ln(a)}{\ln(b)}, \frac{\varphi}{2\pi}, 1\right\}$ is linearly independent over $\mathbb{Z}$.*

Proof. Suppose the graph of ρ is dense in $\mathbb{C}$. Suppose k_1, k_2, k_3 are integers such that

$$k_1\frac{\ln(a)}{\ln(b)} + k_2\frac{\varphi}{2\pi} = k_3.$$

We will show that $k_1 = k_2 = k_3 = 0$. If $k_1 \neq 0$, then let $y_1 = \frac{k_1}{\ln(b)}$ and let $y_2 = \frac{k_2}{2\pi}$. In particular, letting $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, we have $\mathbf{y} \in \mathbb{R}^2 \setminus \{0\}$. Then

$$\begin{bmatrix} \ln(a) & \varphi \\ \ln(b) & 0 \\ 0 & 2\pi \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} k_3 \\ k_1 \\ k_2 \end{bmatrix}.$$

Thus, we have found $\mathbf{y} \in \mathbb{R}^2 \setminus \{0\}$ such that $B^T\mathbf{y} \in \mathbb{Z}^3$. By Corollary 4.9, the graph of ρ is not dense in $\mathbb{C}$, a contradiction. A similar argument shows that if $k_2 \neq 0$, then our assumption that the graph of ρ is dense in $\mathbb{C}$ is contradicted. So we must have $k_1 = k_2 = 0$. Therefore, $k_3 = 0$.

Now suppose the graph of ρ is not dense in $\mathbb{C}$. By Corollary 4.9, there is $\mathbf{y} \in \mathbb{R}^2 \setminus \{0\}$ such that $B^T\mathbf{y} \in \mathbb{Z}^3$. Let $y_1, y_2 \in \mathbb{R}$ and $k_1, k_2, k_3 \in \mathbb{Z}$ be such that

$$y_1 \ln(a) + y_2\varphi = k_3$$

$$y_1 \ln(b) = k_1$$

$$y_2(2\pi) = k_2.$$

From these equations we see that $y_2 = \frac{k_2}{2\pi}$ and $y_1 = \frac{k_1}{\ln(b)}$, so

$$k_1\frac{\ln(a)}{\ln(b)} + k_2\frac{\varphi}{2\pi} = k_3.$$

Since $\mathbf{y} \neq 0$, at least one of y_1, y_2 is not 0. If $y_1 \neq 0$, then $k_1 \neq 0$, so $\left\{\frac{\ln(a)}{\ln(b)}, \frac{\varphi}{2\pi}, 1\right\}$ is linearly dependent over $\mathbb{Z}$. Similarly, if $y_2 \neq 0$, then $k_2 \neq 0$, so $\left\{\frac{\ln(a)}{\ln(b)}, \frac{\varphi}{2\pi}, 1\right\}$ is linearly dependent over $\mathbb{Z}$. $\square$

Corollary 4.11. *The graph of ρ is dense in $\mathbb{C}$ if and only if $(ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}}$ is dense in $\mathbb{C}$.*

Proof. The matrix A defined in Section 4.1 is B with $\psi = 0$. Thus, we have the following equivalences:

$$\begin{aligned} \text{graph}(\rho) \text{ is dense in } \mathbb{C} &\Leftrightarrow \text{ for all } y \in \mathbb{R}^2 \setminus \{0\}, B^T y \notin \mathbb{Z}^3 \quad \text{(Corollary 4.9)} \\ &\Leftrightarrow \text{ the group } (ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}} \text{ is dense in } \mathbb{C} \quad \text{(Corollary 4.4)} \end{aligned}$$

□

If the graph of ρ is dense in $\mathbb{C}$, then by Corollary 4.10, $\frac{\ln(a)}{\ln(b)} \notin \mathbb{Q}$ and $\frac{\varphi}{2\pi} \notin \mathbb{Q}$. In particular, if the graph of ρ is dense in $\mathbb{C}$, then by Proposition 4.5, $a^{\mathbb{Z}}b^{\mathbb{Z}}$ is dense in $\mathbb{R}^{>0}$. Likewise, by Proposition 4.6, $(e^{i\varphi})^{\mathbb{Z}}$ is dense in $\mathbb{S}^1$.

Note that even if $(e^{i\varphi})^{\mathbb{Z}}$ is dense in $\mathbb{S}^1$ and $a^{\mathbb{Z}}b^{\mathbb{Z}}$ is dense in $\mathbb{R}^{>0}$, the group $(ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}}$ does not have to be dense in $\mathbb{C}$. For example, let $\varphi \in \mathbb{R} \setminus 2\pi\mathbb{Q}$ and let $a = e^{\varphi}$. Let $b = e^{(1+i)2\pi}$. (In particular, $b \in \mathbb{R}$.) We have $\ln(a) = \varphi$ and $\ln(b) = 2\pi$, so by Proposition 4.5, $a^{\mathbb{Z}}b^{\mathbb{Z}}$ is dense in $\mathbb{R}^{>0}$. Since $\varphi \notin 2\pi\mathbb{Q}$, by Proposition 4.6, $(e^{i\varphi})^{\mathbb{Z}}$ is dense in $\mathbb{S}^1$. However, $(ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}}$ is a subgroup of the logarithmic spiral $e^{(1+i)\mathbb{R}}$, which is not dense in $\mathbb{C}$.

4.3 Logarithmic spirals

Let $a, b \in \mathbb{R}$ with $a, b > 1$ and let $\varphi \in \mathbb{R}$ be such that $\frac{\ln(a)}{\ln(b)} \notin \mathbb{Q}$ and $\frac{\varphi}{2\pi} \notin \mathbb{Q}$. In this section, we will prove a result relating the density of the group $(ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}}$ in $\mathbb{C}$ to logarithmic spirals.

To prove this result, we first need the following calculation. Let $K_1, K_2, K_3 \in \mathbb{R}$ be such that $K_1 \neq 0$ and $2\pi K_2 + \varphi K_3 \neq 0$. It is not hard to show that

$$2\pi K_1 \left(\frac{\ln(a)}{2\pi K_2 + \varphi K_3} \right) - \ln(b) = 0 \Leftrightarrow K_1 \frac{\ln(a)}{\ln(b)} = K_2 + K_3 \frac{\varphi}{2\pi}. \tag{$*$}$$

Lemma 4.12. *Suppose $\frac{\ln(a)}{\ln(b)} \notin \mathbb{Q}$ and $\frac{\varphi}{2\pi} \notin \mathbb{Q}$. The group $(ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}}$ is not dense in $\mathbb{C}$ if and only if there are nonzero $\omega \in \mathbb{R}$ and $k \in \mathbb{Z}$ such that $a^k e^{ik\varphi}, b^k$ both lie in S_ω.*

Proof. Suppose we have nonzero $\omega \in \mathbb{R}$, $k \in \mathbb{Z}$ such that $a^k e^{ik\varphi}, b^k$ lie in the group $e^{(i+\omega)\mathbb{R}}$. Then there are $t, s \in \mathbb{R}$ such that $a^k e^{ik\varphi} = e^{(i+\omega)t}$ and $b^k = e^{(i+\omega)s}$. Since we assume that $a, b \in \mathbb{R}^{>0}$, we have $t \neq 0$ and $s \neq 0$.

Thus, there are $\ell_1, \ell_2 \in \mathbb{Z}$ such that

$$k\ln(a) + ik\varphi + 2\pi i \ell_1 = it + \omega t \tag{4.5}$$

and

$$k\ln(b) + 2\pi i\ell_2 = is + \omega s. \tag{4.6}$$

Eq. (4.5) gives us that $k\ln(a) = \omega t$ and $k\varphi + 2\pi\ell_1 = t$. Thus,

$$\omega = \frac{k\ln(a)}{t} = \frac{k\ln(a)}{k\varphi + 2\pi\ell_1}.$$

Eq. (4.6) gives us that $k\ln(b) = \omega s$ and $2\pi\ell_2 = s$. Therefore,

$$k\ln(b) = \omega(2\pi\ell_2) = \frac{k\ln(a)}{k\varphi + 2\pi\ell_1}(2\pi\ell_2).$$

We can divide both sides by k to get

$$2\pi\ell_2\left(\frac{\ln(a)}{k\varphi + 2\pi\ell_1}\right) - \ln(b) = 0.$$

Since $k \neq 0$, by $(*)$, $\left\{\frac{\ln(a)}{\ln(b)}, \frac{\varphi}{2\pi}, 1\right\}$ is linearly dependent over $\mathbb{Z}$. By Corollary 4.10, $(ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}}$ is not dense in $\mathbb{C}$.

Now suppose that $(ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}}$ is not dense in $\mathbb{C}$. By Corollary 4.10 there are $K_1, K_2, K_3 \in \mathbb{Z}$ such that

$$K_1\frac{\ln(a)}{\ln(b)} = K_2 + K_3\frac{\varphi}{2\pi}$$

and K_1, K_2, K_3 are not all 0. Since we assume $\frac{\varphi}{2\pi} \notin \mathbb{Q}$, we have $K_1 \neq 0$. Similarly, since we assume $\frac{\ln(a)}{\ln(b)} \notin \mathbb{Q}$, we have $K_3 \neq 0$.

By $(*)$, since $K_1 \neq 0$,

$$\frac{\ln(a)}{2\pi K_2 + K_3\varphi} = \frac{\ln(b)}{2\pi K_1}.$$

Therefore,

$$\frac{\ln(a)}{\ln(b)} = \frac{2\pi K_2 + K_3\varphi}{2\pi K_1}.$$

We want to find $\omega \in \mathbb{R}$, $k \in \mathbb{Z}$ such that for some $s, t \in \mathbb{R}$, $e^{it+\omega t} = a^k e^{ik\varphi}$ and $e^{is+\omega s} = b^k$. Let $t = 2\pi K_2 + K_3\varphi$ and $s = 2\pi K_1$, and let

$$\omega = \frac{K_3\ln(a)}{t} = \frac{K_3\ln(b)}{s}.$$

Since $K_3 \neq 0$, we also have $\omega \neq 0$. We have $t = \dfrac{K_3 \ln(a)}{\omega}$ and $s = \dfrac{K_3 \ln(b)}{\omega}$. By definition of t and ω,

$$e^{it+\omega t} = e^{i(2\pi K_2+\varphi K_3)+K_3 \ln(a)} = a^{K_3} e^{iK_3\varphi}.$$

A similar calculation shows that $e^{is+\omega s} = b^{K_3}$. Therefore, the group generated by $a^{K_3}e^{iK_3\varphi}$ and b^{K_3} lies in $e^{(i+\omega)\mathbb{R}}$ for this value of ω. $\square$

Proposition 4.13. *Let $H = (ae^{i\varphi})^{\mathbb{Z}} b^{\mathbb{Z}}$, where $\frac{\ln(a)}{\ln(b)} \notin \mathbb{Q}$ and $\frac{\varphi}{2\pi} \notin \mathbb{Q}$. If H is not dense in $\mathbb{C}$, then there is nonzero $\omega \in \mathbb{R}$ such that $(\overline{\mathbb{R}}, H)$ defines S_ω.*

Proof. Suppose H is not dense. Then by Lemma 4.12, there are nonzero $\omega \in \mathbb{R}$, $k \in \mathbb{Z}$ such that $H^{[k]} \subseteq S_\omega$. To prove that $(\overline{\mathbb{R}}, H)$ defines S_ω, it suffices to show that $H^{[k]}$ is dense in S_ω. For if we can show this, then S_ω will be the closure of $H^{[k]}$, minus the origin.

Let $f : \mathbb{R} \to S_\omega$ be the function $t \mapsto e^{(i+\omega)t}$. We claim that $f^{-1}(H^{[k]})$ is dense in $\mathbb{R}$, which (since f is continuous and maps onto S_ω) gives us that $H^{[k]}$ is dense in S_ω. We have

$$\begin{aligned} t \in f^{-1}(H^{[k]}) &\Leftrightarrow \text{there exist } j, \ell \in \mathbb{Z} \text{ such that } e^{(i+\omega)t} = e^{jk\ln(a)+jk(i\varphi)} e^{\ell k \ln(b)} \\ &\Leftrightarrow \text{there exist } j, \ell \in \mathbb{Z} \text{ such that } \omega t = jk\ln(a) + \ell k \ln(b). \end{aligned}$$

Therefore,

$$f^{-1}(H^{[k]}) = \frac{1}{\omega}((k\ln(b))\mathbb{Z} + (k\ln(a))\mathbb{Z}).$$

But $\frac{1}{\omega}((k\ln(b))\mathbb{Z} + (k\ln(a))\mathbb{Z})$ is the image of $(a^k)^{\mathbb{Z}}(b^k)^{\mathbb{Z}}$ under the map $x \mapsto \frac{1}{\omega}\ln(x)$. Since we assume $\frac{\ln(a)}{\ln(b)} \notin \mathbb{Q}$, $\frac{k\ln(a)}{k\ln(b)} \notin \mathbb{Q}$. By Proposition 4.5, $(a^k)^{\mathbb{Z}}(b^k)^{\mathbb{Z}}$ is dense in $\mathbb{R}^{>0}$. Since $\frac{1}{\omega}\ln(x)$ is continuous and maps onto $\mathbb{R}$, $f^{-1}(H^{[k]})$ is dense in $\mathbb{R}$.

Therefore, $H^{[k]}$ is dense in S_ω. $\square$

Note that if $(e^{i\varphi})^{\mathbb{Z}}$ is not dense in $\mathbb{S}^1$ and $a^{\mathbb{Z}}b^{\mathbb{Z}}$ is dense in $\mathbb{R}^{>0}$, then Lemma 4.12 fails. For if $(e^{i\varphi})^{\mathbb{Z}}$ is not dense in $\mathbb{S}^1$, then it is finite, with $\left|(e^{i\varphi})^{\mathbb{Z}}\right| = n$ for some nonzero $n \in \mathbb{N}$. Consider the set $\{a^{kn}b^l : k, l \in \mathbb{Z}\}$, which is the subgroup of $(ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}}$ generated by a^n and b. Since we assume that $a^{\mathbb{Z}}b^{\mathbb{Z}}$ is dense in $\mathbb{R}^{>0}$, we have $\frac{\ln(a)}{\ln(b)} \notin \mathbb{Q}$ by Proposition 4.5. Therefore, we also have $\frac{n\ln(a)}{\ln(b)} \notin \mathbb{Q}$, so $a^{n\mathbb{Z}}b^{\mathbb{Z}}$ is also dense in $\mathbb{R}^{>0}$. Similarly, for any nonzero $m \in \mathbb{Z}$, $a^{m\mathbb{Z}}b^{m\mathbb{Z}}$ is dense in $\mathbb{R}^{>0}$. But for any $m \in \mathbb{Z}$, $a^{m\mathbb{Z}}b^{m\mathbb{Z}} = (ae^{i\varphi})^{m\mathbb{Z}}b^{m\mathbb{Z}} \cap \mathbb{R}^{>0}$. Since $S_\omega \cap \mathbb{R}^{>0}$ is not dense in $\mathbb{R}^{>0}$ for any $\omega \in \mathbb{R}$, there cannot exist $\omega \in \mathbb{R}$ and $m \in \mathbb{Z}$ such that $(ae^{i\varphi})^m b^m \in S_\omega$.

In the case where $(e^{i\varphi})^{\mathbb{Z}}$ is not dense in $\mathbb{S}^1$ and $a^{\mathbb{Z}}b^{\mathbb{Z}}$ is dense in $\mathbb{R}^{>0}$, the group $(ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}}$ looks like a star in $\mathbb{C}$. For example, if $\varphi = \frac{\pi}{3}$ and $\frac{\ln(a)}{\ln(b)} \notin \mathbb{Q}$, then the group $(ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}}$ looks like

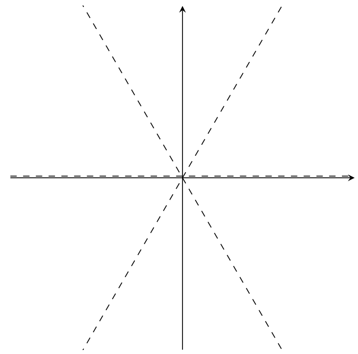

Figure 4.2: The group $(ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}}$, where $\varphi = \frac{\pi}{3}$ and $a^{\mathbb{Z}}b^{\mathbb{Z}}$ is dense in $\mathbb{R}^{>0}$..

We will discuss expansions of $\overline{\mathbb{R}}$ by groups of the form $(ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}}$ further in Chapter 6.

If we make an additional number-theoretic assumption, then we have other conditions for the density of groups of the form $(ae^{i\varphi})^{\mathbb{Z}}(be^{i\psi})^{\mathbb{Z}}$. The phrasing of the following conjecture is taken from Waldschmidt [24].

Four exponentials conjecture. *Let x_1, x_2 be two $\mathbb{Q}$-linearly independent complex numbers and y_1, y_2 also be two $\mathbb{Q}$-linearly independent complex numbers; then at least one of the four numbers*

$$e^{x_1 y_1}, e^{x_1 y_2}, e^{x_2 y_1}, e^{x_2 y_2}$$

is transcendental.

Suppose that the four exponentials conjecture holds. Let $(ae^{i\varphi})^{\mathbb{Z}}(be^{i\psi})^{\mathbb{Z}}$ be a subgroup of $\mathbb{C}$ such that $a, b > 1$, $\dfrac{\ln(a)}{\ln(b)} \notin \mathbb{Q}$, $\varphi \in \mathbb{R} \setminus \pi\mathbb{Q}$ or $\psi \in \mathbb{R} \setminus \pi\mathbb{Q}$, and $ae^{i\varphi}$, $be^{i\psi}$ are algebraic. By the remarks at the bottom of page 171 in [24], under these assumptions, $(ae^{i\varphi})^{\mathbb{Z}}(be^{i\psi})^{\mathbb{Z}}$ is dense in $\mathbb{C}$. In particular, if $\dfrac{\ln(a)}{\ln(b)} \notin \mathbb{Q}$, $\varphi \in \mathbb{R} \setminus \pi\mathbb{Q}$, and $ae^{i\varphi}, b$ are algebraic, then $(\overline{\mathbb{R}}, (ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}})$ cannot define a spiral of the form S_ω for $\omega \neq 0$.

Chapter 5

Subgroups of $\mathbb{C}^{\times}$ generated by a positive real number and an element of the unit circle

In this section, we study expansions of $\overline{\mathbb{R}}$ by a subgroup of $\mathbb{C}$ generated by a real number $a > 1$ and an element $e^{i\varphi} \in \mathbb{S}^1$, where $\varphi \notin \pi\mathbb{Q}$. The following picture shows what such a subgroup looks like in $\mathbb{C}$.

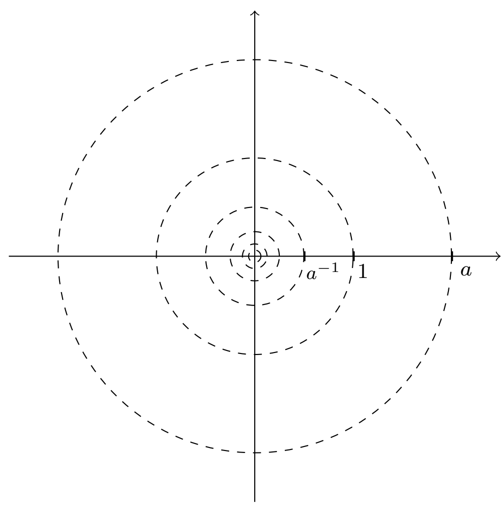

Figure 5.1: The group $a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}}$. This group is dense in each circle with radius in $a^{\mathbb{Z}}$ (represented by the dashed lines).

For the rest of this section, we will fix an infinite finite rank multiplicative subgroup Γ of $\mathbb{S}^1$ and a multiplicative subgroup Δ of $\mathbb{R}^{>0}$ of the form $\varepsilon^{\mathbb{Z}}$ for some $\varepsilon > 1$ in $\mathbb{R}$.

Our first goal in this section is to prove Theorem A. In order to prove this theorem, rather than considering the structure $(\overline{\mathbb{R}}, a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}})$, we instead consider $(\overline{\mathbb{R}}, a^{\mathbb{Z}}, (e^{i\varphi})^{\mathbb{Z}})$. These structures are interdefinable, as $a^{\mathbb{Z}} = a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}} \cap \mathbb{R}^{>0}$ and $(e^{i\varphi})^{\mathbb{Z}} = \{z \in a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}}) : |z| = 1\}$. The reason we split $a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}}$ in this way is so that we can use techniques similar to those used in Chapter 6 of Günaydın [17]. In this chapter, Günaydın studies expansions $(\overline{\mathbb{R}}, \Lambda, \Delta)$ of $\overline{\mathbb{R}}$ by a dense multiplicative subgroup Λ of $\mathbb{R}^{>0}$ and a discrete multiplicative subgroup of Δ of $\mathbb{R}^{>0}$ is presented. The methods used there can be adjusted to work when Λ is replaced by a subgroup Γ of $\mathbb{S}^1$.

5.1 An axiomatization

The first step in proving Theorem A is to obtain an axiomatization for $(\overline{\mathbb{R}}, a^{\mathbb{Z}}, (e^{i\varphi})^{\mathbb{Z}})$ when we add constants for each element of $a^{\mathbb{Z}}$ and of $(e^{i\varphi})^{\mathbb{Z}}$. The theory of this structure can be axiomatized as follows.

Theorem 5.1. *Let K be a real closed field. Let G be a dense subgroup of $\mathbb{S}^1(K)$ and let $\gamma \mapsto \gamma' : \Gamma \to G$ be a group homomorphism. For $\gamma \in \Gamma$ with $\gamma = (\alpha, \beta)$, let α' and β' be such that $\gamma' = (\alpha', \beta')$. Let A be a subgroup of $K^{>0}$ with a group homomorphism $\delta \mapsto \delta' : \Delta \to A$ such that*

(i) ε' is the smallest element of A greater than 1, and

(ii) for every $k \in K^{>0}$, there is $a \in A$ such that $a \leq k < a\varepsilon'$.

Then

$$(K, G, A, (\delta')_{\delta\in\Delta}, (\gamma')_{\gamma\in\Gamma}) \equiv (\overline{\mathbb{R}}, \Gamma, \Delta, (\delta)_{\delta\in\Delta}, (\gamma)_{\gamma\in\Gamma})$$

if and only if:

1. *for every $\gamma \in \Gamma$ and $n \geq 1$, γ is an nth power in Γ if and only if γ' is an nth power in G;*
2. *for all $n \geq 1$, $[n]\Gamma = [n]G$;*
3. *for all $n \geq 1$, all polynomials $Q(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$, and all tuples $(\gamma_1\delta_1, \ldots, \gamma_n\delta_n)$ of elements of $\Gamma\Delta$,*
$$Q(Re(\gamma_1\delta_1), \ldots, Re(\gamma_n\delta_n)) > 0 \text{ if and only if } Q(Re(\gamma_1'\delta_1'), \ldots, Re(\gamma_n'\delta_n')) > 0;$$
4. *$(K, GA, (\gamma'\delta')_{\gamma\in\Gamma,\delta\in\Delta})$ satisfies the Mann axioms for $\Gamma\Delta$;*
5. *all torsion points of G are in Γ.*

In order to prove this theorem, we will construct a back-and-forth system between models of a theory T which satisfy (i) and (ii) in the axiomatization above. Thus, we now give the definition of T. Let $\mathcal{L}_{or}(P, V, \Gamma, \Delta)$ be the language consisting of $\mathcal{L}_{or}$, the language of ordered rings, together with a binary predicate symbol P, unary predicate symbol V, and constants for each element of $\mathrm{Re}(\Gamma) \cup \mathrm{Im}(\Gamma) \cup \Delta$.

For $\gamma \in \Gamma$ with $\gamma = (\alpha, \beta)$, let $\gamma' := (\alpha', \beta')$. Although $\mathcal{L}$-structures have constants for each element of $\mathrm{Re}(\Gamma) \cup \mathrm{Im}(\Gamma) \cup \Delta$, we will write $\mathcal{L}$-structures in the form

$$(K, G, A, (\gamma')_{\gamma\in\Gamma}, (\delta')_{\delta\in\Delta})$$

for convenience.

Definition 5.2. Let T_1 be the $\mathcal{L}_{or}(P, V, \Gamma, \Delta)$-theory whose models have the form

$$(K, G, A, (\gamma')_{\gamma\in\Gamma}, (\delta')_{\delta\in\Delta})$$

such that:

1. K is a real closed field
2. A is a subgroup of $K^{>0}$ such that ε' is the smallest element of A larger than 1
3. G is a dense subgroup of $\mathbb{S}^1(K)$
4. for all $x \in K^{>0}$, there is $a \in A$ such that $a \leq x < a\varepsilon$
5. $\delta \mapsto \delta' : \Delta \to A$ and $\gamma \mapsto \gamma'$: $\Gamma \to G$ are group homomorphisms
6. $(K, (\gamma'\delta')_{\gamma\in\Gamma,\delta\in\Delta})$ satisfies the orientation axioms for $\Gamma\Delta$.
7. $(K, GA, (\gamma'\delta')_{\gamma\in\Gamma,\delta\in\Delta})$ satisfies the Mann axioms for $\Gamma\Delta$
8. $G_{\text{tor}} = \Gamma_{\text{tor}}$

Here A is a multiplicative subgroup of the real closed field K with a smallest element larger than 1. Moreover, we assume that for all $x \in K^{>0}$, there is $a \in A$ such that $a \leq x < a\varepsilon$. Therefore, by Lemma 2.12, A is regularly discrete.

By Lemma 2.10, for each $n > 0$ and each $a \in A$, there is $i \in \{1, \ldots, n\}$ such that a is congruent to $(\varepsilon')^i$ modulo $A^{[n]}$.

For convenience, we will identify the subgroup Γ' of K^2 with Γ and the subgroup Δ' of K with Δ. Thus, we will write γ rather than γ' and δ rather than δ'.

Let $\mathcal{M} := (K, G, A, (\gamma)_{\gamma\in\Gamma}, (\delta)_{\delta\in\Delta})$ be a model of T_1. We will use the following lemma in Section 5.1.2.

Lemma 5.3. *Let $f_1, \ldots, f_n : (GA)^m \to K$ be functions ($n, m \geq 1$) which are definable in the language $\mathcal{L}_{or}(K)$. Then $K \setminus \bigcup_{j=1}^n f_j((GA)^m)$ is dense in K.*

Proof. This follows directly from Corollary 2.10 in [18] once we show that GA is small in K. Since $\mathcal{M} \models T$, GA has the Mann property. Therefore, by Proposition 1.1 in van den Dries and Günaydın [13] and Proposition 2.9 in [18], GA is small in $K(i)$. By Lemma 2.8 in [18], GA is small in K. $\square$

5.1.1 Substructures of models of T_1

Let $\mathcal{M} := (K, G, A, (\gamma)_{\gamma\in\Gamma}, (\delta)_{\delta\in\Delta})$ be a model of T_1. Let κ be an infinite cardinal such that $\kappa > |\Gamma\Delta|$.

We can consider A as a subgroup of $K(i)^\times$ by identifying the element a of A with the element $a + 0i$ of $K(i)$. Letting 1 denote the identity of G, we have $G \cap A = \{1\}$ when A is considered as a subgroup of $K(i)^\times$.

Definition 5.4. Let $\mathcal{S}(K, G, A)$ be the collection of $\mathcal{L}_{or} \cup \{P, V\}$-structures (K', G', A') such that:

1. K' is a real closed subfield of K of cardinality less than κ
2. G' is a pure subgroup of G containing Γ
3. A' is a pure subgroup of A containing Δ
4. $K'(i)$ and $\mathbb{Q}(GA)$ are free over $\mathbb{Q}(G'A')$
5. For all $k \in (K')^{>0}$, there is $a \in A'$ such that $a \le k < a\varepsilon$.

Note that in particular, we require that $G' \subseteq K'(i)$ and $A' \subseteq K'$.

Lemma 5.5. *If (K', G', A') satisfies conditions (1)-(4) in Definition 5.4, then (K', G', A') is indeed a substructure of (K, G, A). Moreover, if (K', G', A') satisfies conditions (1)-(4) in Definition 5.4, then satisfying condition (5) is equivalent to K' being closed under λ.*

Proof of Lemma 5.5. Suppose (K', G', A') satisfies conditions (1)-(4). We want to show that $G \cap K'(i) = G'$ and $A \cap K' = A'$. Since G' is a subgroup of G, it is clear that $G' \subseteq G \cap K'(i)$. Now let $g \in K'(i) \cap G$. Then in particular, g is algebraic over $\mathbb{Q}(G)$, so g is algebraic over $\mathbb{Q}(GA)$. By condition (4) in Definition 5.4, $K'(i)$ and $\mathbb{Q}(GA)$ are free over $\mathbb{Q}(G'A')$, so g is algebraic over $\mathbb{Q}(G'A')$. Let $p(x) \in \mathbb{Q}(G'A')[x]$ be a polynomial such that $p(g) = 0$ and let $d = \deg(p)$. By assumption, $\mathcal{M} \models T$, so in particular, $(K, GA, (\gamma\delta)_{\gamma\in\Gamma,\delta\in\Delta})$ satisfies the Mann axioms for $\Gamma\Delta$. Therefore, we may apply Lemma 5.12 in [13] to conclude that $g^d \in G'A'$. Thus, there are $h \in G', a \in A'$ such that $g^d = ha$, so $a = h^{-1}g^d$. Since $G \cap A = \{1\}$, we must have $a = 1$, so $g^d = h$. By purity of G' in G (condition (2) in Definition 5.4), there is $g' \in G'$ such that $g^d = h = (g')^d$. Since $\mathcal{M} \models T$, in particular, $G_{\text{tor}} = \Gamma_{\text{tor}}$; thus, there is $\gamma \in \Gamma$ such that $g = g'\gamma$. Since condition (2) of Definition 5.4 gives us that $\Gamma \subseteq G'$, we have $g \in G'$.

The proof that $A \cap K' = A'$ is similar, using condition (3) in Definition 5.4. Note that if $a^d = (a')^d$ for $a', a \in A$, we automatically have $a = a'$. This is because $A \subseteq K^{>0}$ and K is a real closed field.

Now suppose (K', G', A') satisfies conditions (1)-(4) in Definition 5.4. Since (K', G', A') satisfies conditions (1)-(4), (K', G', A') is a substructure of (K, G, A). Therefore, $(K') \cap A = A'$; moreover, since $A' \subseteq (K')^{>0}$, $(K')^{>0} \cap A = A'$.

Suppose that in addition, (K', G', A') satisfies condition (5) in Definition 5.4. We want to show that K' is closed under λ; that is, we want to show that $\lambda((K')^{>0}) = K' \cap A = A'$. Let $k \in (K')^{>0}$. By condition (5) in Definition 5.4, there is $a' \in A'$ such that $a' \leq k < a'\varepsilon$. By definition of λ, we have $\lambda(k) = a'$, so $\lambda(k) \in A'$. Conversely, let $a' \in A'$. We must have $\lambda(a') = a'$ by definition of λ, so it is clear that $A' \subseteq \lambda((K')^{>0})$. Therefore, K' is closed under λ.

Conversely, suppose that in addition to (K', G', A') satisfying conditions (1)-(4) in Definition 5.4, K' is closed under λ. Let $k' \in (K')^{>0}$. Since we assume K' is closed under λ, we have $\lambda((K')^{>0}) = (K')^{>0} \cap A = A'$. Therefore, $\lambda(k') \in A'$, so $\lambda(k')$ is an element of A' such that $\lambda(k') \leq k' < \lambda(k')\varepsilon$. Thus, condition (5) of Definition 5.4 is fulfilled. $\square$

Note that if (K', G', A') is an $\mathcal{L}_{or}(P, V)$-structure satisfying conditions (1) and (5) in Definition 5.4, then A' is pure in A. However, in practice, to show that an $\mathcal{L}_{or}(P, V)$-structure (K', G', A') is in $\mathcal{S}(K, G, A)$, we will first check conditions (1)-(4) to show that (K', G', A') is a substructure of (K, G, A), then show that K' is closed under λ.

5.1.2 Elementary equivalence of models of T_1

In this section we establish Theorem 5.1. In fact, we prove the following slightly more general result.

Theorem 5.6. *Let $(K, G, A, (\gamma)_{\gamma\in\Gamma}, (\delta)_{\delta\in\Delta})$ and $(L, H, B, (\gamma)_{\gamma\in\Gamma}, (\delta)_{\delta\in\Delta})$ be two models of T_1. Then they are elementarily equivalent if and only if $[n]G = [n]H$ for all $n \geq 1$, and for all $\gamma \in \Gamma$ and all $n \geq 1$:*

$$\gamma \text{ is an } n\text{th power in } G \text{ if and only if } \gamma \text{ is an } n\text{th power in } H.$$

First suppose $(K, G, A, (\gamma)_{\gamma\in\Gamma}, (\delta)_{\delta\in\Delta}) \equiv (L, H, B, (\gamma)_{\gamma\in\Gamma}, (\delta)_{\delta\in\Delta})$. The statements "$[n]G = m$" and "$\gamma$ is/is not an nth power in G" are first-order sentences in our language. We can also express the statement "$[n]G = \infty$" using first-order sentences in our language. Thus, the "only if" direction of the theorem statement is clear.

We now prove the other direction of Theorem 5.6. Fix two models

$$\mathcal{M} := (K, G, A, (\gamma)_{\gamma\in\Gamma}, (\delta)_{\delta\in\Delta}) \text{ and } \mathcal{N} := (L, H, B, (\gamma)_{\gamma\in\Gamma}, (\delta)_{\delta\in\Delta})$$

of T_1 such that $[n]G = [n]H$ for all $n \geq 1$, and for all $\gamma \in \Gamma$ and all $n \geq 1$:

$$\gamma \text{ is an } n\text{th power in } G \text{ if and only if } \gamma \text{ is an } n\text{th power in } H.$$

We want to prove that $\mathcal{M} \equiv \mathcal{N}$. We may assume that $\mathcal{M}, \mathcal{N}$ are κ-saturated for some infinite $\kappa > |\Gamma\Delta|$. Let $\mathcal{I}$ be the collection of isomorphisms between members of $\mathcal{S}(K, G, A)$ and $\mathcal{S}(L, H, B)$ that fix Δ and Γ pointwise. We will show that $\mathcal{I}$ is a nonempty back-and-forth system, which will give us that $\mathcal{M} \equiv \mathcal{N}$.

As stated at the end of Section 2.5, whenever we apply the Fundamental Lemma below, we will take $\mu : K \to K(i)$ to be the function which maps every element of K to 1.

$\mathcal{I}$ is nonempty

To see that $\mathcal{I}$ is nonempty, let

$$K' = \mathbb{Q}(\mathrm{Re}(\Gamma\Delta))^{\mathrm{rc}}, G' = \{g \in G : g^n \in \Gamma \text{ for some } n > 0\}, A' = \Delta$$

and let

$$L' = \mathbb{Q}(\mathrm{Re}(\Gamma\Delta))^{\mathrm{rc}}, H' = \{h \in H : h^n \in \Gamma \text{ for some } n > 0\}, B' = \Delta.$$

We must first check that $K' \subseteq K$, $G' \subseteq (K')^2$, and $A' \subseteq K'$. It is clear that $K' \subseteq K$. If $g \in G'$, then there is $n > 0$ such that $g^n \in \Gamma$. Let $z = g^n$, $a = \mathrm{Re}(z)$, and $b = \mathrm{Im}(z)$. Since $z \in \Gamma$, $a^2 + b^2 = 1$, so it can be checked that $z^2 - 2az + 1 = 0$. Therefore, g is algebraic over $\mathbb{Q}(\mathrm{Re}(\Gamma\Delta))^{\mathrm{rc}}$, and so $g \in K'(i)$.

We now check that $(K', G', A') \in \mathcal{S}(K, G, A)$. By Lemma 2.18, A' is pure in A. We apply Corollary 2.8 with $X = \emptyset$ and $k = \mathbb{Q}(\mathrm{Re}(\Gamma\Delta))$ to get that $K'(i)$ and $\mathbb{Q}(GA)$ are free over $\mathbb{Q}(G'A')$. Now we must show that K' is closed under λ. Since (K', G', A') satisfies conditions (1)-(4) of Definition 5.4, $K' \cap A = A'$. By Lemma 2.18 and Lemma 2.17, $\lambda((K')^{>0}) = \Delta = A'$, so K' is closed under λ.

The proof that $(L', H', B') \in \mathcal{S}(L, H, B)$ is similar.

We now show that there is a function $f : (K', G', A') \to (L', H', B')$ with $f \in \mathcal{I}$. To prove this, let $f : K' \to L'$ be the natural function extending the identity map on $\mathrm{Re}(\Gamma\Delta)$. Let p_1 be the set of $\mathcal{L}_{or}$-formulas satisfied by elements of $\mathrm{Re}(\Gamma\Delta)$ in $\mathcal{M}$, and let p_2 be the set of $\mathcal{L}_{or}$-formulas satisfied by elements of $\mathrm{Re}(\Gamma\Delta)$ in $\mathcal{N}$. Since we assume that $\mathcal{M}$ and $\mathcal{N}$ satisfy the orientation axioms for $\Gamma\Delta$, we have $p_1 = p_2$ by Lemma 3.10. Therefore, f is an ordered field isomorphism. Clearly, $f(A') = B'$. Similarly, by our assumption that γ is an nth power in G if and only if γ is an nth power in H, $f(G') = H'$. (Let $g : K^2 \to L^2$ be the function defined by $g(k_1, k_2) = (f(k_1), f(k_2))$. By $f(G')$, we mean $g(G')$.) Clearly f fixes $\Gamma\Delta$ pointwise.

Therefore, $\mathcal{I}$ is nonempty.

$\mathcal{I}$ is a back-and-forth system

Let $(K', G', A') \in \mathcal{S}(K, G, A)$ and $(L', H', B') \in \mathcal{S}(L, H, B)$. Let $\iota : (K', G', A') \to (L', H', B')$ be in $\mathcal{I}$, and let $a \in K \setminus K'$. We have four cases:

1. $a \in A$
2. $a \in \mathrm{Re}(G)$ or $a \in \mathrm{Im}(G)$
3. $a \in K'(\mathrm{Re}(GA) \cup \mathrm{Im}(GA))^{\mathrm{rc}}$
4. $a \in K \setminus K'(\mathrm{Re}(GA) \cup \mathrm{Im}(GA))^{rc}$

Case 1. $a \in A$.

Define sets Σ_1, Σ_2 of $\mathcal{L}_{om}(V, K')$-formulas in the variable x by

$$\Sigma_1 := \{\iota(k_1) < x < \iota(k_2) : a \in (k_1, k_2), k_1, k_2 \in K'\},$$

$$\Sigma_2 := \{\iota(a')x^l \in B^{[m]} : a' \in A', l \in \mathbb{Z}, m > 0, a'a^l \in A^{[m]}\}.$$

Our first step is to find $b \in B$ such that b satisfies $\Sigma_1 \cup \Sigma_2$.

Since $(K', G', A') \in \mathcal{S}(K, G, A)$, A' is pure in A. Therefore, $[p]A' \leq [p]A$ for all primes p. Moreover, since $(K', G', A') \in \mathcal{S}(K, G, A)$, $\Delta \subseteq A'$. Therefore, by the last axiom in T_1, $[p]A' \geq [p]A$. So we have $[p]A' = [p]A$ for all primes p. Similarly, we have $[p]B' = [p]B$ for all primes p. Since A and B are regularly discrete, we have $[p]A = [p]B = p$ for all primes p by Lemma 2.10. Since $\mathcal{M}, \mathcal{N}$ are κ-saturated (where $\kappa > |K'|$), (A, A') and (B, B') are κ-saturated. Therefore, we may apply Lemma 4.2.1 in [17]. Using this lemma, we fix $h \in B$ such that for all $a' \in A'$, $m > 0$, and $l \in \mathbb{Z}$,

$$a'a^l \in A^{[m]} \text{ if and only if } \iota(a')h^l \in B^{[m]}.$$

In order to prove that $\Sigma_1 \cup \Sigma_2$ is satisfiable by an element of B, we show that $\Sigma_1 \cup \Sigma_2$ is finitely satisfiable by an element of B. For if we can show this, then $\Sigma_1 \cup \Sigma_2$ is satisfied by an element $b \in B$ by κ-saturation of (L, H, B).

To prove that $\Sigma_1 \cup \Sigma_2$ is finitely satisfiable by an element of B, it suffices to show that for given $k_1, k_2 \in K'$ such that $k_1 < a < k_2$, there exists $\beta \in B$ such that $\iota(k_1) < \beta < \iota(k_2)$ and β satisfies Σ_2. Thus, we fix $k_1, k_2 \in K'$ such that $k_1 < a < k_2$. We may assume that $k_1, k_2 > 0$ since $a \in A \subseteq K^{>0}$.

Since K' is closed under λ and $a \notin A'$, we have

$$k_1 < \lambda(k_1)\varepsilon < \lambda(k_1)\varepsilon^2 < \ldots < a < k_2.$$

Therefore, the interval contains infinitely many elements of A'.

Since $\iota : (K', G', A') \to (L', H', B')$ is an isomorphism, the interval $(\iota(k_1), \iota(k_2))$ in L' contains infinitely many elements of B'. Since $h \in B$, the interval $I := (\iota(k_1)h^{-1}, \iota(k_2)h^{-1})$ contains infinitely many elements of B.

Consider the set of $\mathcal{L}_{or}(V, B)$-formulas

$$\{x \in (\iota(k_1)h^{-1}, \iota(k_2)h^{-1}) \wedge \exists y \in B(x = y^k) : k > 0\}.$$

By κ-saturation, to find an element satisfying this set of formulas, it suffices to find an element $x \in I \cap \bigcap_{i=1}^{s} B^{[n_i]}$ for arbitrary $n_1, \ldots, n_s \geq 1$ and $s \geq 1$. If we are given $n_1, \ldots, n_s$, let $n = n_1 \ldots n_s$. Since there are infinitely many elements of B in I, let b_1, b_2 be elements of $I \cap B$ such that there are at least n elements of B in the interval (b_1, b_2). Since B is assumed to be regularly discrete, there is $\eta' \in B'$ with $\eta' \in (b_1, b_2) \cap B^{[n]}$ by Lemma 2.10. Therefore, $\eta' \in I \cap \bigcap_{i=1}^{s} B^{[n_i]}$.

Let $\eta \in I \cap \bigcap_{k \geq 1} B^{[k]}$. In particular, η is divisible in B by all $k \geq 1$. Let $\beta = h\eta^{1/n}$. Note that $h^n\eta \in (\iota(k_1), \iota(k_2))$, so $\beta^n \in (\iota(k_1), \iota(k_2))$.

It follows from our choice of β that for all $a' \in A', l \in \mathbb{Z}$, and $m > 0$, we have

$$a'a^l \in A^{[m]} \text{ if and only if } \iota(a')\beta^l \in B^{[m]}.$$

Therefore, $\iota(k_1) < \beta < \iota(k_2)$ and β satisfies Σ_2.

We now have $b \in B$ such that $\mathcal{N} \models \Sigma_1(b) \cup \Sigma_2(b)$. Since b satisfies the same cut over K' that a does over L', we have an $\mathcal{L}_{or}$-isomorphism $\iota' : K'(a)^{\mathrm{rc}} \to L'(b)^{\mathrm{rc}}$ extending ι which takes a to b. Since b satisfies Σ_2, we have $\iota'(A'\langle a\rangle_A) = B'\langle b\rangle_B$. To check that $\iota' \in \mathcal{I}$, we must check that

$$(K'(a)^{\mathrm{rc}}, G', A'\langle a\rangle_A) \in \mathcal{S}(K, G, A).$$

In particular, we must show that $(K'(a)^{\mathrm{rc}})(i)$ and $\mathbb{Q}(GA)$ are free over $\mathbb{Q}(G'A'\langle a\rangle_A)$. By assumption, $K'(i)$ and $\mathbb{Q}(GA)$ are free over $\mathbb{Q}(G'A')$. Therefore, we may apply Lemma 2.7 with $E = \{a\}$ and $X = \emptyset$ to get that $K'(a)^{\mathrm{rc}}(i)$ and $\mathbb{Q}(GA)$ are free over $\mathbb{Q}(G'A', a)$. Since $\mathbb{Q}(G'A', a) \subseteq \mathbb{Q}(G'A'\langle a\rangle_A)$, $K'(i)$ and $\mathbb{Q}(GA)$

are free over $\mathbb{Q}(G'A'\langle a\rangle_A)$.

We now want to prove that $K'(a)^{\mathrm{rc}}$ is closed under λ. Since $(K'(a)^{\mathrm{rc}}, G', A'\langle a\rangle_A)$ is a substructure of (K, G, A), it suffices to prove that $\lambda((K'(a)^{\mathrm{rc}})^{>0}) = A'\langle a\rangle_A$. Since $a \in A$, in order to prove this, it suffices to prove that $v(a) \notin v(K')$ by the Fundamental Lemma. Suppose for a contradiction that $v(a) \in v(K')$. By the proof of the Fundamental Lemma, if $v(a) \in v(K')$, then $a \in K'$. But by assumption, $a \in K \setminus K'$, a contradiction. So we must have $v(a) \notin v(K')$.

The proof that $(L'(b)^{\mathrm{rc}}, H', B'\langle b\rangle_B) \in \mathcal{S}(L, H, B)$ is similar.

Case 2. Now suppose $a \in \mathrm{Re}(G)$. (The case where $a \in \mathrm{Im}(G)$ is similar.)

Let $A^{(1)} := A'$ and for $j = 1, 2, \ldots$, let $A^{(j+1)} = \lambda((K'(A^{(j)}, a)^{\mathrm{rc}})^{>0})$. Let $A^{\infty} := \bigcup_{j=1}^{\infty} A^{(j)}$. Note that $A^{(j)} \subseteq A^{(j+1)}$ for all j by definition. Moreover, for each j, $\left|A^{(j)}\right| < \kappa$ since $\kappa > |K'|$. Therefore, $|A^{\infty}| < \kappa$.

Let $B^{(1)} := B'$. For $j \geq 1$, we recursively define $B^{(j)} \subseteq B$ and an ordered field isomorphism $f_j : K'(A^{(j)})^{\mathrm{rc}} \to L'(B^{(j)})^{\mathrm{rc}}$ such that $f_j \in \mathcal{I}$ and $f_j(A^{(j)}) = B^{(j)}$. In particular, we require that $(K'(A^{(j)})^{\mathrm{rc}}, G', A^{(j)}) \in \mathcal{S}(K, G, A)$ and $(L'(B^{(j)})^{\mathrm{rc}}, H', B^{(j)}) \in \mathcal{S}(L, H, B)$ for all j. Note that $K'(A^{(1)}) = K'$ and $L'(B^{(1)}) = B'$. Thus, we take $f_1 : K'(A^{(1)})^{\mathrm{rc}} \to L'(B^{(1)})^{\mathrm{rc}}$ to be ι. Now suppose we have defined f_j and $B^{(j)}$ $(j \geq 1)$, and we want to define $B^{(j+1)}$ and f_{j+1}.

Let $K_j = K'(A^{(j)})^{\mathrm{rc}}$ and let $L_j = L'(B^{(j)})^{\mathrm{rc}}$. Since $f_j \in \mathcal{I}$, K_j is closed under λ. Therefore, $\lambda(K_j^{>0}) = A^{(j)}$. To define $B^{(j+1)}$ and f_{j+1}, we consider two cases: (1) for all $x \in (K_j(a)^{\mathrm{rc}})^{>0}$, $v(x) \in v(K_j)$, and (2) there is $z \in (K_j(a)^{\mathrm{rc}})^{>0}$ such that $v(z) \notin v(K_j)$.

First assume that for all $x \in (K_j(a)^{\mathrm{rc}})^{>0}$, $v(x) \in v(K_j)$. By Case 1 of the Fundamental Lemma, $A^{(j+1)} = \lambda((K_j(a)^{\mathrm{rc}})^{>0}) = A^{(j)}$. So we take $B^{(j+1)} = B^{(j)}$ and $f_{j+1} = f_j$.

Now suppose that there is $z \in (K_j(a)^{\mathrm{rc}})^{>0}$ such that $v(z) \notin v(K_j)$.

We show that in this case, $A^{(j+1)} = A^{(j)}\langle\lambda(f(a))\rangle_A$ for some $\mathcal{L}_{or}(K_j)$-definable function f. Let f be an $\mathcal{L}_{or}(K_j)$-definable function such that $z = f(a)$. Then by Case 2 of the Fundamental Lemma, we have $\lambda((K_j(a)^{\mathrm{rc}})^{>0}) = A^{(j)}\langle\lambda(f(a))\rangle_A$. (Here we must use the assumption that $\lambda(K_j^{>0}) = A^{(j)}$.) But since $K_j = K'(A^{(j)})^{\mathrm{rc}}$, we have $K_j(a)^{\mathrm{rc}} = K'(A^{(j)}, a)^{\mathrm{rc}}$. Therefore,

$$A^{(j+1)} = \lambda((K'(A^{(j)}, a)^{\mathrm{rc}})^{>0}) = \lambda((K_j(a)^{\mathrm{rc}})^{>0}) = A^{(j)}\langle\lambda(f(a))\rangle_A.$$

By definition, $K_{j+1} = K'(A^{(j+1)})^{\mathrm{rc}}$. Thus, by what we just proved, $K_{j+1} = K'(A^{(j)}\langle\lambda(f(a))\rangle_A)^{\mathrm{rc}}$. It can be shown that

$$K'(A^{(j)}\langle\lambda(f(a))\rangle_A)^{\mathrm{rc}} = (K'(A^{(j)})(\lambda(f(a))))^{\mathrm{rc}}$$

so $K_{j+1} = K_j(\lambda(f(a)))^{\mathrm{rc}}$.

By our inductive assumption, $f_j : (K_j, G', A^{(j)}) \to (L_j, H', B^{(j)})$ is in $\mathcal{I}$. Moreover, $\lambda(f(a)) \in A$ and $\lambda(f(a)) \notin K_j$. (Since $v(f(a)) = v(\lambda(f(a))$ and we assume $v(f(a)) \notin v(K_j)$, we cannot have $\lambda(f(a)) \in K_j$.) Therefore, we may apply Case 1 of this theorem to find $b \in B$ and an ordered field isomorphism

$$f_{j+1} : (K_j(\lambda(f(a)))^{\mathrm{rc}}, G', A^{(j)}\langle\lambda(f(a))\rangle_A) \to (L_j(b)^{\mathrm{rc}}, H', B^{(j)}\langle b\rangle_B)$$

with $f_{j+1} \in \mathcal{I}$ taking $\lambda(f(a))$ to b. Thus, in this case, we take $B^{(j+1)}$ to be $B^{(j)}\langle b\rangle_B$. Note that f_{j+1} extends f_j by construction.

This completes the recursive construction. Now define

$$f_\infty := \bigcup_{j\geq 1} f_j, B^\infty := \bigcup_{j\geq 1} B^{(j)}.$$

We will now show that $(K'(A^\infty)^{\mathrm{rc}}, G', A^\infty) \in \mathcal{S}(K, G, A)$, $(L'(B^\infty)^{\mathrm{rc}}, H', B^\infty) \in \mathcal{S}(L, H, B)$, and $f_\infty \in \mathcal{I}$.

To show that $(K'(A^\infty)^{\mathrm{rc}}, G', A^\infty) \in \mathcal{S}(K, G, A)$, we first show that A^∞ contains Δ and is pure in A. Since $A' \subseteq A^\infty$ and A' contains Δ by assumption, A^∞ also contains Δ. The pureness of A^∞ follows easily from the pureness of $A^{(N)}$ for each N.

We now check the freeness condition. By assumption, $K'(i)$ and $\mathbb{Q}(GA)$ are free over $\mathbb{Q}(G'A')$. By definition of A^∞, we have $A' \subseteq A^\infty$. Therefore, we may apply Lemma 2.7 with $E = A^\infty$ and $X = \emptyset$ to show that $K'(A^\infty)^{\mathrm{rc}}(i)$ and $\mathbb{Q}(GA)$ are free over $\mathbb{Q}(G'A^\infty)$.

We now want to check that $K'(A^\infty)^{\mathrm{rc}}$ is closed under λ. In particular, we must show that $\lambda((K'(A^\infty)^{\mathrm{rc}})^{>0}) \subseteq A^\infty$. Let $x \in (K'(A^\infty)^{\mathrm{rc}})^{>0}$. Then $x \in (K'(A^{(N+1)})^{\mathrm{rc}}$ for some $N \in \mathbb{N}$. By construction, we have $\lambda(x) \in A^{(N+2)}$. Therefore, $\lambda(x) \in A^\infty$.

The proof that $(L'(B^\infty)^{\mathrm{rc}}, H', B^\infty) \in \mathcal{S}(L, H, B)$ is similar, using the construction of B^∞. By construction, f_∞ is an ordered field isomorphism between $K'(A^\infty)^{\mathrm{rc}}$ and $L'(B^\infty)^{\mathrm{rc}}$ taking A^∞ to B^∞. Since f_∞ extends ι, it fixes Γ and Δ, so $f_\infty \in \mathcal{I}$.

Our next step is to find $\iota' \in \mathcal{I}$ such that ι' extends f_∞ and a is in the domain of ι'. Since $a \in \mathrm{Re}(G)$, let $g \in G$ with $a = \mathrm{Re}(g)$. By assumption, G and H are regularly dense oriented abelian groups and $[p]G = [p]H$ for all primes p. Since $\mathcal{M}, \mathcal{N} \models T$, we also have $G_{\mathrm{tor}} = G'_{\mathrm{tor}}$ and $H_{\mathrm{tor}} = H'_{\mathrm{tor}}$. Moreover, G' is pure in G and H' is pure in H. Thus, we apply Lemma 2.23 to obtain $\eta \in H$ and an oriented group isomorphism $j : G'\langle g\rangle_G \to H'\langle\eta\rangle_H$ taking g to η and extending f_∞. (That is, for $(\alpha, \beta) \in G'$, $j(\alpha, \beta) = (f_\infty(\alpha), f_\infty(\beta))$.)

We now find $h \in H$ such that h satisfies the set of $\mathcal{L}_{or}(L', P)$-formulas $S_1 \cup S_2 \cup S_3$ in the variable x, where

$$S_1 = \{f_\infty(k_1) < \mathrm{Re}(x) < f_\infty(k_2) : k_1 < \mathrm{Re}(g) < k_2, k_1, k_2 \in K'(A^\infty)^{\mathrm{rc}}\},$$

$$S_2 = \{f_\infty(g')x^l \in H^{[m]} : g'g^l \in G^{[m]}, g' \in G', l \in \mathbb{Z}, m > 0\},$$

$$S_3 = \{f_\infty(g')x^l \notin H^{[m]} : g'g^l \notin G^{[m]}, g' \in G', l \in \mathbb{Z}, m > 0\}$$

In order to find an element of H satisfying $S_1 \cup S_2 \cup S_3$, it suffices by κ-saturation of $\mathcal{N}$ to show that every finite subset of $S_1 \cup S_2 \cup S_3$ is realized by an element of H. As in Case 1, it suffices to find $z \in H$ realizing a single formula from S_1 such that z also satisfies S_2 and S_3.

Thus, let $k_1, k_2 \in K'(A^\infty)^{\mathrm{rc}}$ be such that $k_1 < \mathrm{Re}(g) < k_2$. We may assume without loss of generality that $k_1, k_2 \in [-1, 1]$. Let $y_1, y_2 \in K$ such that $y_1 = (1 - (f_\infty(k_1))^2)^{1/2}$, $y_2 = (1 - (f_\infty(k_2))^2)^{1/2}$. (In particular, $y_1, y_2 > 0$.) Let $z_1 = (f_\infty(k_1), y_1)$ and let $z_2 = (f_\infty(k_2), y_2)$. Thus, y_1, y_2 are elements of $L'(B^\infty)^{\mathrm{rc}}$ such that $z_1, z_2 \in \mathbb{S}^1(L)$. Since $k_1 < k_2$, $\mathcal{O}(1, z_2, z_1)$ holds in $\mathcal{M}$. Let

$$I = \{z \in \mathbb{S}^1(L) : \mathcal{M} \models \mathcal{O}(z_2\eta^{-1}, z, z_1\eta^{-1})\}.$$

That is, I is the "interval" in $\mathbb{S}^1(L)$ between $z_2\eta^{-1}$ and $z_1\eta^{-1}$.

We claim that there is $z \in \bigcap_{m=1}^\infty H^{[m]}$ such that $z \in I$. By κ-saturation of $\mathcal{N}$, it suffices to find $z' \in I$ with $z' \in \bigcap_{j=1}^n H^{[m_j]}$ for arbitrary $m_1, \ldots, m_n \geq 1$, $n \geq 1$. If we are given $m_1, \ldots, m_n$, let $m = m_1 \ldots m_n$. By regular density of H in $\mathbb{S}^1(L)$, there is $z' \in H^{[m]}$ such that $z' \in I$. Since $m = m_1 \ldots m_n$, we also have $z \in H^{[m_j]}$ for $1 \leq j \leq n$.

Thus, let z be an element of I with $z \in \bigcap_{m=1}^\infty H^{[m]}$. By definition of I, $\mathcal{N} \models \mathcal{O}(z_2, z\eta, z_1)$. In particular, $f_\infty(k_1) < \mathrm{Re}(z\eta) < f_\infty(k_2)$ holds.

We now show that for all $g' \in G'$, $l \in \mathbb{Z}$, $m > 0$ such that $g'g^l \in G^{[m]}$, we have $f_\infty(g')(z\eta)^l \in H^{[m]}$. By our choice of z, we have $z \in H^{[m]}$, so we also have $z^l \in H^{[m]}$. Since j is an oriented group isomorphism extending f_∞ and taking g to η, we have $f_\infty(g')\eta^l \in H^{[m]}$. Therefore, $f_\infty(g')(z\eta)^l \in H^{[m]}$.

We must also show that for all $g' \in G'$, $l \in \mathbb{Z}$, $m > 0$ such that $g'g^l \notin G^{[m]}$, we have $f_\infty(g')(z\eta)^l \notin H^{[m]}$. Suppose $f_\infty(g')(z\eta)^l \in H^{[m]}$. By our choice of z, we have $z^{-l} \in H^{[m]}$. Therefore, $f_\infty(g')\eta^l \in H^{[m]}$. Since j is an oriented group isomorphism extending f_∞ and taking g to η, we have $g'g^l \in G^{[m]}$.

So $z\eta$ satisfies every formula in S_2 and S_3, as well as the formula $f_\infty(k_1) < \mathrm{Re}(z\eta) < f_\infty(k_2)$ for our given k_1, k_2.

By κ-saturation of $\mathcal{N}$, we have $h \in H$ such that h satisfies $S_1 \cup S_2 \cup S_3$. Since $\mathrm{Re}(h)$ satisfies the same cut over $L'(B^\infty)$ that a does over $K'(A^\infty)$, we can extend f_∞ to an ordered field isomorphism $\iota' : K'(A^\infty, a)^{\mathrm{rc}} \to$

$L'(B^\infty, \mathrm{Re}(h))^{\mathrm{rc}}$ taking a to $\mathrm{Re}(h)$. Moreover, since ι' is an ordered field isomorphism, $\iota'(g) = h$. Thus, we have $\iota'(G'\langle g\rangle_G) = H'\langle h\rangle_H$ by our choice of h. Since ι' extends f_∞, we also have $\iota'(A^\infty) = B^\infty$.

We now show that $(K'(A^\infty, a)^{\mathrm{rc}}, G'\langle g\rangle_G, A^\infty) \in \mathcal{S}(K, G, A)$. We first check the freeness condition. As proved above, $K'(A^\infty)^{\mathrm{rc}}$ and $\mathbb{Q}(GA)$ are free over $\mathbb{Q}(G'A^\infty)$. Therefore, we apply Lemma 2.7 with $E = \{g\}$ and $X = \emptyset$ to get that $K'(A^\infty, a)^{\mathrm{rc}}(i)$ and $\mathbb{Q}(GA)$ are free over $\mathbb{Q}(G'A^\infty, g)$. Since $\mathbb{Q}(G'A^\infty, g) \subseteq \mathbb{Q}(G'\langle g\rangle_G A^\infty)$, we see that $K'(A^\infty, a)^{\mathrm{rc}}(i)$ and $\mathbb{Q}(GA)$ are free over $\mathbb{Q}(G'\langle g\rangle_G A^\infty)$.

Next, we want to show that $\lambda((K'(A^\infty, a)^{\mathrm{rc}})^{>0}) = A^\infty$. Let $x \in (K'(A^\infty, a)^{\mathrm{rc}})^{>0}$. Then $x = \sigma(k, c, a)$ for some $\mathcal{L}_{or}$-definable function σ, some tuple k of elements of elements of K', and some tuple c of elements of A^∞. Since c is a tuple of elements from A^∞, we must have $c \subseteq A^{(j)}$ for some $j \geq 1$. We have $\lambda(\sigma(k, c, a)) \in A^{(j+1)}$ by definition of $A^{(j+1)}$, so $\lambda(x) \in A^\infty$.

Let $b = \mathrm{Re}(h)$. The proof that $(L'(B^\infty, b)^{\mathrm{rc}}, H'\langle h\rangle_H, B^\infty) \in \mathcal{S}(L, H, B)$ is mostly similar to the proof that $(K'(A^\infty, a)^{\mathrm{rc}}, G'\langle g\rangle_G, A^\infty) \in \mathcal{S}(K, G, A)$. We only need to show that $L'(B^\infty, b)^{\mathrm{rc}}$ is closed under λ.

Let $\sigma(\ell, d, b) \in (L'(B^\infty, b)^{\mathrm{rc}})^{>0}$, where σ is an $\mathcal{L}_{or}$-definable function, ℓ is a tuple of elements from L', and d is a tuple of elements from B^∞. Let $y = \lambda(\sigma((\iota')^{-1}(\ell), (\iota')^{-1}(d), a))$, so that $y \in A^\infty$. By definition of λ,

$$y \leq \sigma((\iota')^{-1}(\ell), (\iota')^{-1}(d), a) \leq \varepsilon y.$$

Since ι' is an isomorphism taking a to b and fixing Δ, $\iota'(y) \leq \sigma(\ell, d, b) \leq \varepsilon\iota'(y)$. Therefore, $\lambda(x) = \iota'(y)$. Since $\iota'(A^\infty) = B^\infty$, we have $\lambda(x) \in B^\infty$. Therefore, $L'(B^\infty, b)^{\mathrm{rc}}$ is closed under λ.

Therefore, ι' is an element of $\mathcal{I}$ extending ι with a in its domain.

Case 3. Suppose $a \in K'(\mathrm{Re}(GA) \cup \mathrm{Im}(GA))^{\mathrm{rc}}$.

Since $a \in K'(\mathrm{Re}(GA) \cup \mathrm{Im}(GA))^{\mathrm{rc}}$, there are tuples $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ of elements of G, tuples $e = (e_1, \ldots, e_n)$ and $c = (c_1, \ldots, c_m)$ of elements of A, and an $\mathcal{L}_{or}(K')$-definable function σ such that

$$a = \sigma(\mathrm{Re}(x_1)e_1, \ldots, \mathrm{Re}(x_n)e_n, \mathrm{Im}(y_1)c_1, \ldots, \mathrm{Im}(y_m)c_m)$$

By using Case 1 repeatedly, we find tuples $b = (b_1, \ldots, b_n)$ and $d = (d_1, \ldots, d_m)$ of elements of B and an isomorphism

$$\iota' : (K'(e, c)^{\mathrm{rc}}, G', A'\langle e, c\rangle_A) \to (L'(b, d)^{\mathrm{rc}}, H', B'\langle b, d\rangle_B)$$

extending ι with $\iota' \in \mathcal{I}$.

Now let $K'' := K'(e, c)^{\mathrm{rc}}$ and $L'' := L'(b, d)^{\mathrm{rc}}$. By using Case 2 repeatedly, we find $w_1, \ldots, w_n$,

$z_1, \dots, z_m \in H$, $A'' \subseteq A$, $B'' \subseteq B$, and an isomorphism

$$\iota'' : (K''(A'', \mathrm{Re}(x), \mathrm{Im}(y))^{\mathrm{rc}}, G'\langle x, y\rangle_G, A'') \to (L''(B'', \mathrm{Re}(w), \mathrm{Im}(z))^{\mathrm{rc}}, H'\langle w, z\rangle_H, B'')$$

extending ι' with $\iota'' \in \mathcal{I}$. In particular, $A'\langle e, c\rangle_A \subseteq A''$ and $B'\langle b, d\rangle_B \subseteq B''$. Note that a is in the domain of ι''. Thus, $\iota'' \in \mathcal{I}$ extends ι and has a in its domain.

Case 4. Suppose $a \in K \setminus K'(\mathrm{Re}(GA) \cup \mathrm{Im}(GA))^{\mathrm{rc}}$.

As in Case 2 above, we first extend ι to an isomorphism

$$f_\infty : (K'(A^\infty)^{\mathrm{rc}}, G', A^\infty) \to (L'(B^\infty)^{\mathrm{rc}}, H', B^\infty)$$

where $A^\infty, B^\infty, f_\infty$ are defined as in Case 2.

Next, we want to find $b \in L \setminus L'(\mathrm{Re}(HB) \cup \mathrm{Im}(HB))^{\mathrm{rc}}$ such that b realizes the same cut over $L'(B^\infty)^{\mathrm{rc}}$ that a does over $K'(A^\infty)^{\mathrm{rc}}$. We will then extend f_∞ to an element of $\mathcal{I}$ that maps a to b.

Let Φ_1 be the collection of formulas of the form

$$\neg(\exists h = (h_1, \dots, h_n) \in H^n \exists b = (b_1, \dots, b_n) \in B^n [x = f(p_1(h_1 b_1), \dots, p_n(h_n b_n))])$$

where f is a $\mathcal{L}_{or}(L')$-definable function from L^n to L and each p_j is either Re or Im. Thus, if φ is a formula in Φ_1 of the above form, there is an $\mathcal{L}_{or}(L')$-definable function $f_\varphi : (HB)^n \to L$ such that

$$f_\varphi(h_1 b_1, \dots, h_n b_n) = f(p_1(h_1 b_1), \dots, p_n(h_n b_n))$$

for all $h_1, \dots, h_n \in H$ and $b_1, \dots, b_n \in B$. Let Φ_2 be the collection of formulas

$$\Phi_2 := \{f_\infty(k_1) < x < f_\infty(k_2) : k_1, k_2 \in K'(A^\infty)^{\mathrm{rc}}, k_1 < a < k_2\}.$$

If we have finitely many formulas $\varphi_1, \dots, \varphi_k$ in Φ_1, there are $s_1, \dots, s_k \in \mathbb{N}$ and functions $f_{\varphi_1}, \dots, f_{\varphi_k}$ such that for each $j \in \{1, \dots, k\}$, $f_{\varphi_j} : (HB)^{s_j} \to L$ is a $\mathcal{L}_{or}(L')$-definable function. We can assume that there is $m \in \mathbb{N}$ such that each f_{φ_j} is a function from $(HB)^m$ to L. By Lemma 5.3, the set $L \setminus \bigcup_{j=1}^k f_j(HB)^m)$ is dense in L. Therefore, given a finite subset of formulas $\Phi' \subseteq \Phi_1 \cup \Phi_2$, we can find $x \in L$ satisfying Φ'.

By κ-saturation of $\mathcal{N}$, there is b that satisfies all formulas in $\Phi_1 \cup \Phi_2$. This b lies in $L \setminus L'(\mathrm{Re}(HB) \cup \mathrm{Im}(HB))^{\mathrm{rc}}$ and realizes the same cut over $L'(B^\infty)^{\mathrm{rc}}$ that a does over $K'(A^\infty)^{\mathrm{rc}}$. Therefore, there is an ordered field isomorphism $\iota' : K'(A^\infty, a)^{\mathrm{rc}} \to L'(B^\infty, b)^{\mathrm{rc}}$ extending f_∞.

We check that $(K'(A^\infty, a)^{\mathrm{rc}}, G', A^\infty) \in \mathcal{S}(K, G, A)$. In particular, we must show that $K'(A^\infty, a)^{\mathrm{rc}}(i)$ and $\mathbb{Q}(GA)$ are free over $\mathbb{Q}(G'A^\infty)$. First note that since $K'(\mathrm{Re}(GA))^{\mathrm{rc}}(i)$ is algebraically closed and $a \in K \setminus K'(\mathrm{Re}(GA))^{\mathrm{rc}}$, a must be algebraically independent over $K'(\mathrm{Re}(GA))^{\mathrm{rc}}(i)$. Since $K'(GA) \subseteq K'(\mathrm{Re}(GA))^{\mathrm{rc}}(i)$, a is also algebraically independent over $K'(GA)$. By construction, $K'(A^\infty)^{\mathrm{rc}}$ and $\mathbb{Q}(GA)$ are free over $\mathbb{Q}(G'A^\infty)$. Therefore, we may apply Lemma 2.7 with $E = \emptyset$ and $X = \{a\}$ to get that $K'(A^\infty, a)^{\mathrm{rc}}(i)$ and $\mathbb{Q}(GA)$ are free over $\mathbb{Q}(G'A^\infty)$. We must also show that $K'(A^\infty, a)^{\mathrm{rc}}$ is closed under λ. But this follows by definition of A^∞ as in Case 2.

This completes the proof that $\mathcal{I}$ is a back-and-forth system.

5.2 Predicate-near model completeness

In this section we will give a proof of Theorem A. We start by introducing the notion of a special formula and show that T_1 has quantifier elimination up to Boolean combinations of these formulas. Note that T_1 is not complete and makes no assumptions on the cardinality of $[n]G$ in a model of $(K, G, A, (\gamma)_{\gamma\in\Gamma}, (\delta)_{\delta\in\Delta})$ of T_1. Adding the requirement that $[n]G$ is finite for each n, we will establish the following stronger theorem.

Theorem 5.7. *Let $\mathcal{M} := (K, G, A, (\gamma)_{\gamma\in\Gamma}, (\delta)_{\delta\in\Delta})$ be a model of T_1 such that $[n]G$ is finite for each n. Then every subset of K^m definable in $\mathcal{M}$ is a boolean combination of subsets of K^m defined in $\mathcal{M}$ by formulas of the form*

$$\exists y \exists z (V(y) \wedge P(z) \wedge \phi(x, y, z))$$

where $\phi(x, y, z)$ is a quantifier free $\mathcal{L}_{or}(K)$-formula.

By Theorem 2.4, every finite rank subgroup of $\mathbb{S}^1(\mathbb{R})$ satisfies the assumptions of Theorem 5.7.

5.2.1 Special formulas and types

In order to prove Theorem A, we first define the notion of a special formula and show that our theory T_1 eliminates quantifiers up to special formulas. We then use this to prove Theorem A.

We first give the definition of the V-restriction of a formula. This definition is taken from page 10 of [17].

Definition 5.8. Let $\mathcal{L}$ be a language and let V be a unary relation symbol not in $\mathcal{L}$. Let $\mathcal{L}(V)$ be the language $\mathcal{L}$ augmented by V. The V-restriction of ϕ is the $\mathcal{L}(V)$-formula ϕ_V defined recursively as follows.

- If ϕ is atomic, then $\phi_V := \phi$.

- If $\phi = \neg\phi'$, then $\phi_V := \neg\phi'_V$.
- If $\phi = \phi' \wedge \phi''$, then $\phi_V := \phi'_V \wedge \phi''_V$.
- If $\phi = \phi' \vee \phi''$, then $\phi_V := \phi'_V \vee \phi''_V$.
- If $\phi = \exists x\phi'$, then $\phi_V := \exists x(V(x) \wedge \phi'_V)$.
- If $\phi = \forall x\phi'$, then $\phi_V := \forall x(V(x) \rightarrow \phi'_V)$.

A *special* $\mathcal{L}_{or}(P, V, \Gamma, \Delta)$*-formula* in x (where x is a tuple of variables) with parameters from S is a formula $\psi(x)$ of the form

$$\exists y \exists z(V(y) \wedge P(z) \wedge \theta^1_V(y) \wedge \theta^2_P(z) \wedge \phi(x, y, z))$$

where y is a tuple of variables, z is a tuple of pairs of variables, $\theta^1(y)$ is an $\mathcal{L}_{om}(\Delta)$-formula, $\theta^2(z)$ is an element of $\Sigma_{\mathrm{orm}}(\Gamma)$ (as defined in Section 2.4), $\theta^1_V(y)$ is the V-restriction of $\theta^1(y)$, $\theta^2_P(z)$ is the P-restriction of $\theta^2(z)$, and $\phi(x, y, z)$ is an $\mathcal{L}_{or}(\Gamma, \Delta, S)$-formula. If $y = (y_1, \ldots, y_n)$ and $z = ((z_{11}, z_{12}), \ldots, (z_{m1}, z_{m2}))$, then $V(y)$ is an abbreviation for $V(y_1) \wedge \ldots \wedge V(y_n)$ and $P(z)$ is an abbreviation for $P(z_{11}, z_{12}) \wedge \ldots \wedge P(z_{m1}, z_{m2})$. By a *special formula (in x)*, we mean a special $\mathcal{L}_{or}(P, V, \Gamma, \Delta)$-formula in x with parameters from $\emptyset$.

Now let $\mathcal{M} := (K, G, A, (\gamma)_{\gamma\in\Gamma}, (\delta)_{\delta\in\Delta})$ be a model of T_1. Let $Y, C \subseteq K$. The *special type of Y over C*, denoted $\mathrm{sptp}^{\mathcal{M}}(Y|C)$, is the set of special formulas with parameters from C satisfied by Y in $\mathcal{M}$.

The following fact is Fact 1 in [13], translated to fit our situation.

Fact 5.9. Let B be the Boolean algebra of T_1-equivalence classes of $\mathcal{L}_{or}(P, V, \Gamma, \Delta)$-formulas in the variables $x = (x_1, \ldots, x_m)$. For an $\mathcal{L}_{or}(P, V, \Gamma, \Delta)$-formula $\phi(x)$, let $\phi(x)/T_1$ denote its T_1-equivalence class. Let $\Psi \subseteq B$ denote the set of (cosets of) special $\mathcal{L}_{or}(P, V, \Gamma, \Delta)$-formulas in x. For an $\mathcal{L}_{or}(P, V, \Gamma, \Delta)$-type $p(x)$ containing T_1, let $[p(x)] = \{\phi(x)/T_1 : \phi(x) \in p(x)\}$. Suppose that for any $p_1, p_2 \in S_x(T_1)$,

$$\text{if } [p_1(x)] \cap \Psi = [p_2(x)] \cap \Psi, \text{ then } [p_1(x)] = [p_2(x)].$$

Then Ψ generates B as a Boolean algebra.

Next we fix some notation that we will use in the rest of this thesis. Let $\mathcal{L}$ be a language and let $\mathcal{A}$ be an $\mathcal{L}$-structure. Whenever $C, Y \subseteq A$, $\mathrm{tp}^{\mathcal{A}}(Y|C)$ will denote the $\mathcal{L}(C)$-type of Y. For a sublanguage $\mathcal{L}'$ of $\mathcal{L}$, $\mathrm{tp}^{\mathcal{A}}_{\mathcal{L}'}(Y|C)$ will denote the $\mathcal{L}'(C)$-type of Y.

Let $\mathcal{B}$ be another $\mathcal{L}$-structure and fix an injective function $f : C \rightarrow B$. We define $f(\mathrm{tp}^{\mathcal{A}}_{\mathcal{L}'}(Y|C))$ by

$$f(\mathrm{tp}^{\mathcal{A}}_{\mathcal{L}'}(Y|C)) = \{\phi(x, f(c)) : \phi(x, z) \text{ an } \mathcal{L}'\text{-formula}, c \in C^{|y|}, \phi(x, c) \in \mathrm{tp}^{\mathcal{A}}_{\mathcal{L}'}(Y|C)\}.$$

If $\mathcal{A}$ is an $\mathcal{L}_{or}(P, V, \Gamma, \Delta)$-structure, then we define $f(\mathrm{sptp}^{\mathcal{A}}(Y|C))$ by

$$f(\mathrm{sptp}^{\mathcal{A}}(Y|C)) = \{\phi(x, f(c)) : \phi(x, z) \text{ a special formula}, c \in C^{|y|}, \phi(x, c) \in \mathrm{sptp}^{\mathcal{A}}(Y|C)\}.$$

5.2.2 Quantifier elimination up to special formulas

In this section, we prove that T_1 eliminates quantifiers up to special formulas.

Lemma 5.10. *Each $\mathcal{L}_{or}(P, V, \Gamma, \Delta)$-formula $\psi(x)$ is equivalent in T_1 to a Boolean combination of special $\mathcal{L}_{or}(P, V, \Gamma, \Delta)$-formulas in x.*

Proof. Let $\kappa > |\Gamma\Delta|$ and let

$$\mathcal{M} := (K, G, A, (\gamma)_{\gamma\in\Gamma}, (\delta)_{\delta\in\Delta}), \mathcal{N} := (L, H, B, (\gamma)_{\gamma\in\Gamma}, (\delta)_{\delta\in\Delta})$$

be κ-saturated models of T_1. Let $\alpha := (\alpha_1, \ldots, \alpha_m) \in K^m$ and $\beta := (\beta_1, \ldots, \beta_m) \in L^m$ satisfy (in $\mathcal{M}$ and $\mathcal{N}$ respectively) the same special formulas in x. By Fact 5.9, to prove the lemma, it suffices to show that $\mathrm{tp}^{\mathcal{M}}(\alpha) = \mathrm{tp}^{\mathcal{N}}(\beta)$. First note that the formulas expressing $[n]G$ and $[n]H$ $(n \geq 1)$ are special formulas. Therefore, $[n]G = [n]H$ for each n. Since $\mathcal{M}, \mathcal{N} \models T_1$, A and B are regularly discrete. Therefore, for each $n \geq 1$, $[n]A = n$ by Lemma 2.10. Moreover, the formula expressing "γ is an nth power in G" is a special formula. Therefore, γ is an nth power in G if and only if γ is an nth power in H for all $\gamma \in \Gamma$. Therefore, we have a back-and-forth system $\mathcal{I}$ between $\mathcal{M}$ and $\mathcal{N}$ as constructed in Theorem 5.6. To show that $\mathrm{tp}^{\mathcal{M}}(\alpha) = \mathrm{tp}^{\mathcal{N}}(\beta)$, we will find $\iota \in \mathcal{I}$ such that each α_j is in the domain of ι and $\iota(\alpha_j) = \beta_j$ for each $j \in \{1, \ldots, m\}$.

Throughout, let $\alpha := (\alpha_1, \ldots, \alpha_m)$ and let $\beta := (\beta_1, \ldots, \beta_m)$. Let $F = \mathbb{Q}(\mathrm{Re}(\Gamma\Delta))$. Let $\mathbb{Q}(GA)(\alpha)$ have transcendence degree r over $\mathbb{Q}(GA)$. We may assume that $\{\alpha_1, \ldots, \alpha_r\}$ is a subset of $\{\alpha_1, \ldots, \alpha_m\}$ that is maximal with respect to being algebraically independent over $\mathbb{Q}(GA)$. Thus, we have a tuple g of elements of G, a tuple a of elements of A, and $\mathcal{L}_{or}$-definable functions $\sigma_{r+1}, \ldots, \sigma_m$ such that for each $j \in \{r+1, \ldots, m\}$,

$$\sigma_j(g, a, \alpha_1, \ldots, \alpha_r) = \alpha_j.$$

By a similar argument as in Theorem 3.8 of [13], using the fact that α and β satisfy the same special formulas, $\{\beta_1, \ldots, \beta_r\}$ is algebraically independent over $\mathbb{Q}(HB)$.

We first define $A^\infty \subseteq A$ in a similar way as in Case 2 of this theorem. That is, let $A^{(1)} = \Delta$ and for

$j \geq 1$, define $K_j = F(A^{(j)})^{\mathrm{rc}}$ and

$$A^{(j+1)} = \lambda(K_j(\alpha, g, a)^{\mathrm{rc}}).$$

Let $A^\infty = \bigcup_{j\geq 1} A^{(j)}$. Note that since $\kappa > |\Gamma\Delta|$, we have $|A^\infty| < \kappa$. Let $|A^\infty| = \rho$. Consider $\mathrm{sptp}^{\mathcal{M}}(g, A^\infty|\alpha)$ as a set of $\mathcal{L}_{or}(P, V, \Gamma, \Delta)$-formulas in the variables $(x_\eta : \eta < \rho)$. We will show that $\mathrm{sptp}^{\mathcal{M}}(g, A^\infty|\alpha)$ is finitely satisfiable in $\mathcal{N}$ when each α_j is replaced by the corresponding β_j. Let $\mu : \{\alpha_1, \ldots, \alpha_m\} \to \{\beta_1, \ldots, \beta_m\}$ be the function defined by $\mu(\alpha_i) = \beta_i$ for each i. Suppose $\phi_1(c, g, \alpha), \ldots, \phi_n(c, g, \alpha)$ are formulas in $\mathrm{sptp}^{\mathcal{M}}(g, A^\infty|\alpha)$, where $\phi_1, \ldots, \phi_n$ are $\mathcal{L}_{or}(P, V, \Gamma, \Delta)$-formulas and c is a tuple of elements of A^∞. Then

$$\phi(\alpha) := \exists y \in V \exists z \in P(\phi_1(y, z, \alpha) \wedge \ldots \wedge \phi_n(y, z, \alpha))$$

is equivalent to a special formula $\psi(\alpha)$ in $\mathrm{sptp}^{\mathcal{M}}(\alpha)$. By our assumption that $\mathrm{sptp}^{\mathcal{M}}(\alpha) = \mathrm{sptp}^{\mathcal{N}}(\beta)$, we have $\psi(\beta) \in \mathrm{sptp}^{\mathcal{N}}(\beta)$. By κ-saturation of $\mathcal{N}$, there is a subset B^∞ of elements of B and tuple h of elements of H such that $\mu(\mathrm{sptp}^{\mathcal{M}}(g, A^\infty|\alpha)) \subseteq \mathrm{sptp}^{\mathcal{N}}(h, B^\infty|\beta)$. In particular,

$$\mu(\mathrm{tp}^{\mathcal{M}}_{\mathcal{L}_{or}}(g, A^\infty|\alpha)) = \mathrm{tp}^{\mathcal{N}}_{\mathcal{L}_{or}}(h, B^\infty|\beta). \tag{$*$}$$

Let $\mathcal{G}$ denote the $\mathcal{L}_{orm}$-structure with universe G, with the orientation and multiplication on G interpreted as in Section 2.4. Let $\mathcal{H}$ denote the $\mathcal{L}_{orm}$-structure with universe H, again with orientation and multiplication interpreted as in Section 2.4. Since $\mu(\mathrm{sptp}^{\mathcal{M}}(g, A^\infty|\alpha)) \subseteq \mathrm{sptp}^{\mathcal{N}}(h, B^\infty|\beta)$, we also have

$$\mathrm{tp}^{\mathcal{G}}(g) = \mathrm{tp}^{\mathcal{H}}(h). \tag{$**$}$$

Note that each equation $\sigma_j(c, g', \alpha_1, \ldots, \alpha_r) = \alpha_j$ (for $j \in \{r+1, \ldots, m\}$ and tuples c of elements from A^∞ and g' of elements from G) corresponds to a special formula in $\mathrm{sptp}^{\mathcal{M}}(g, A^\infty/\alpha)$. Therefore, for $j \in \{r+1, \ldots, m\}$, we also have

$$\sigma_j(d, h', \beta_1, \ldots, \beta_r) = \beta_j$$

for some tuples d of elements from A^∞ and h' of elements from h.

Let

$$K' = F(\alpha, g, A^\infty)^{\mathrm{rc}}, \quad G' = \Gamma\langle g\rangle_G, \quad A' = A^\infty,$$

$$L' = F(\beta, h, B^\infty)^{\mathrm{rc}}, \quad H' = \Gamma\langle h\rangle_H, \quad B' = B^\infty.$$

By $(*)$, we have an ordered field isomorphism $\iota : K' \to L'$ which takes g to h, A^∞ to B^∞, and α to β. We claim that $\iota \in \mathcal{I}$. By construction, $\iota(A^\infty) = B^\infty$. We now want to show that $\iota(G') = H'$. To do this, it suffices to show that for all $\gamma \in \Gamma$, $p_1, \ldots, p_k \in \mathbb{Z}$, and $n > 0$,

$$\gamma g_1^{p_1} \ldots g_k^{p_k} \in G^{[n]} \text{ if and only if } \gamma h_1^{p_1} \ldots h_k^{p_k} \in H^{[n]}.$$

But this follows from $(**)$.

We now show that $(K', G', A') \in \mathcal{S}(K, G, A)$ and $(L', H', B') \in \mathcal{S}(L, H, B)$. In particular, we must show that $K'(i)$ and $\mathbb{Q}(GA)$ are free over $\mathbb{Q}(G'A')$. Let $k = F(\mathrm{Re}(g), A^\infty)$ and let $X = \{\alpha_1, \ldots, \alpha_r\}$. Note that $k \subseteq \mathbb{Q}(\mathrm{Re}(G'A'))$. By our choice of X, X is algebraically independent over $\mathbb{Q}(GA)$. Therefore, we may apply Corollary 2.8 to get that $K'(i)$ and $\mathbb{Q}(GA)$ are free over $\mathbb{Q}(G'A')$. The fact that K' is closed under λ follows by definition of K'. Next we must check that $L'(i)$ and $\mathbb{Q}(HB)$ are free over $\mathbb{Q}(H'B')$. Let $Y = \{\beta_1, \ldots, \beta_r\}$. As stated previously, Y is algebraically independent over $\mathbb{Q}(HB)$. Therefore, a similar proof as before shows that $L'(i)$ and $\mathbb{Q}(HB)$ are free over $\mathbb{Q}(H'B')$. Lastly, we must check that L' is closed under λ. Since $\iota(A^\infty) = B^\infty$, we have $y = \sigma(\beta_1, \ldots, \beta_r, h, \iota(c))$ for some $\mathcal{L}_{or}$-definable function σ and tuple c of elements from A^∞. Consider $x := \lambda(\sigma(\alpha, g, c))$. Then

$$x \leq \sigma(\alpha_1, \ldots, \alpha_r, g, c) < \varepsilon x.$$

Since ι is an ordered field isomorphism which takes g to h and takes α_i to β_i for $i \in \{1, \ldots, r\}$, we have

$$\iota(x) \leq \sigma(\beta_1, \ldots, \beta_r, h, \iota(c)) < \varepsilon\iota(x).$$

Therefore, $\iota(x) = \lambda(y)$. Since $\iota(x) \in B^\infty$, we have $\lambda(y) \in B^\infty$.

Therefore, $\iota \in \mathcal{I}$, $\alpha_1, \ldots, \alpha_m$ are in the domain of $\mathcal{I}$, and $\iota(\alpha_j) = \beta_j$ for each j. This finishes the proof of the lemma. $\square$

5.2.3 Proof of Theorem 5.7

We are now ready to prove Theorem 5.7.

By Lemma 5.10, it suffices to show that subsets of K^m defined by special formulas have the desired form. Let

$$\psi(x) := \exists y \exists z (V(y) \wedge P(z) \wedge \theta_V^1(y) \wedge \theta_P^2(z) \wedge \phi(x, y, z))$$

be a special formula where $y = (y_1, \dots, y_n)$ is a tuple of variables and

$$z = ((z_{11}, z_{12}), \dots, (z_{j1}, z_{j2}))$$

is a tuple of pairs of variables.

By Lemma 3.13 and Lemma 3.14, the set $\{a \in A^n : A \models \theta_V^1(a)\}$ is a Boolean combination of subsets of A^n, each of which has one of the following forms:

$$\{a \in A^n : \delta^k a_1^{k_1} \dots a_n^{k_n} = 1\} \text{ or}$$

$$\{a \in A^n : \delta^k a_1^{k_1} \dots a_n^{k_n} < 1\} \text{ or}$$

$$\{a \in A^n : \delta^k a_1^{k_1} \dots a_n^{k_n} \in A^{[d]}\}$$

where $k_1, \dots, k_n \in \mathbb{Z}$, $\delta \in \Delta$, k is a tuple of elements from $\mathbb{Z}$, and d is a positive integer. Since $\mathcal{M} \models T$, $[d]A$ is finite. Therefore, a set of the form

$$A^n \setminus \{a \in A^n : \delta^k a_1^{k_1} \dots a_n^{k_n} \in A^{[d]}\}$$

is equal to a finite union of sets of the form

$$\{a \in A^n : \delta^k a_1^{k_1} \dots a_n^{k_n} \in a' A^{[d]}\}$$

where $a' \in A$. Therefore, $\psi(x)$ is equivalent in $\mathcal{M}$ to a formula $\psi'(x)$ with

$$\psi'(x) := \exists y' \exists z (V(y') \wedge P(z) \wedge \theta_P^2(z) \wedge \phi'(x, y', z))$$

where y' is a tuple of variables (extending y) and ϕ' is an $\mathcal{L}_{or}(A)$-formula.

Now consider the subgroup $G \subseteq \mathbb{S}^1(K)$. Since we assume $[n]G$ is finite for each $n \geq 1$, there is a function e from the set of prime numbers to $\mathbb{N}$ such that $[p]G = p^{e(p)}$ for each prime p. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the languages defined before Lemma 3.11, and let $\Sigma(e)$ be the set of $\mathcal{L}_2$-sentences defined in Lemma 3.11. We can make G into an $\mathcal{L}_2$-structure $\mathcal{G}$ such that $\mathcal{G} \models \Sigma(e)$ by taking $\mathcal{O}$ to be the orientation on G inherited from $\mathbb{S}^1(K)$. Now note that there is an $\mathcal{L}_2$-formula $\theta(z)$ such that for all $g \in G$, $\mathcal{M} \models \theta_P^2(g)$ if and only if $\mathcal{G} \models \theta(g)$. By Lemma 3.11 and Lemma 3.12, θ is equivalent in $\mathcal{G}$ to a Boolean combination of subsets of G^j, each of which has one of the following forms:

$$\gamma^k x_1^{k_1} \dots x_n^{k_n} = 1 \text{ or}$$

$$\mathcal{O}(\gamma_1^k x_1^{k_1} \dots x_n^{k_n}, \gamma_2^k y_1^{k_1} \dots y_m^{k_m}, \gamma_3^k z_1^{i_1} \dots z_p^{i_p}) \text{ or}$$

$$E_d(\gamma^k x_1^{k_1} \dots x_n^{k_n})$$

where $k_1, \dots, k_n, l_1, \dots, l_m, i_1, \dots, i_p \in \mathbb{Z}$, k, l, i are tuples of elements of $\mathbb{Z}$, d is a positive integer, and $\gamma, \gamma_1, \gamma_2, \gamma_3$ are tuples of elements from Γ. Since we assume $[n]G$ is finite for each $n \geq 1$, a set of the form

$$G^j \setminus \{g \in G^j : \gamma^k g_1^{k_1} \dots g_n^{k_n} \in G^{[d]}\}$$

is equal to a finite union of sets of the form

$$\{g \in G^j : \gamma^k g_1^{k_1} \dots g_n^{k_n} \in g' G^{[d]}\}$$

where $g' \in G$. Therefore, there is an existential $\mathcal{L}_1(G)$-formula $\theta'(z)$ such that for all $g \in G$, $\mathcal{G} \models \theta(g) \leftrightarrow \theta'(g)$. Therefore, $\psi'(x)$ is equivalent in $\mathcal{M}$ to a formula $\psi''(x)$ with

$$\psi''(x) := \exists y' \exists z' (V(y') \wedge P(z') \wedge \phi''(x, y', z'))$$

where z' is a tuple of pairs of variables extending z and ϕ'' is an $\mathcal{L}_{or}(A, G)$-formula.

Note that ϕ'' may not be quantifier free; however, by quantifier elimination for real closed fields, we can find a quantifier free $\mathcal{L}_{or}(K)$-formula $\chi(x, y', z')$ such that

$$\mathcal{M} \models \forall x (\exists y' \exists z' (V(y' \wedge P(z') \wedge \phi''(x, y', z')) \leftrightarrow \exists y' \exists z' (V(y') \wedge P(z') \wedge \chi(x, y', z'))).$$

Therefore, ψ is equivalent in $\mathcal{M}$ to a formula of the desired form.

5.3 Definable open sets

In this section, we prove Theorem B. Let $\mathcal{L}^* = \mathcal{L}_{or}(P, V)$ and let $\mathcal{L} = \mathcal{L}_{or}(V)$. Let $\mathcal{R}^* = (\overline{\mathbb{R}}, \Gamma, \Delta)$ and let $\mathcal{R} = (\overline{\mathbb{R}}, \Delta)$. Using Corollary 3.1 from Boxall and Hieronymi [4], we will show that every open set definable in $\mathcal{R}^*$ is already definable in $\mathcal{R}$. Throughout this section, "open" will mean open in the usual order topology.

Let $\mathcal{M}^*$ be a κ-saturated, strongly κ-homogeneous elementary extension of $\mathcal{R}^*$ (where $\kappa = |\mathbb{R}|^+$) with

$\mathcal{M}^* = (M, G, A)$. Let $\mathcal{M}$ be the reduct of $\mathcal{M}^*$ to $\mathcal{L}$, so that $\mathcal{M} = (M, A)$. Let $\mathcal{N}$ be the reduct of $\mathcal{M}$ to $\mathcal{L}_{or}$. Let C be a countable subset of M. Note A has a smallest element larger than 1 which we again denote by ε. Moreover, since $\Gamma\Delta$ has the Mann property, GA also has the Mann property.

In order to use Corollary 3.1 in [4], we need to check that Assumption (I) in that paper is satisfied. We now state Assumption (I), translated to fit our situation.

Assumption (I): for any $n \geq 1$ and $(a_1, \ldots, a_n) \in U \subseteq M^n$ such that U is open, the set X_U given by

$$X_U := \{(b_{11}, b_{12}, \ldots, b_{n1}, b_{n2}) \in M^{2n} : [b_{11} < a_1 < b_{12}] \wedge \ldots \wedge [b_{n1} < a_n < b_{n2}] \wedge (b_{11}, b_{12}) \times \ldots \times (b_{n1}, b_{n2}) \subseteq U\}$$

has nonempty interior.

Proof that Assumption (I) is satisfied. We prove that Assumption (I) is satisfied for M by induction on n. Since $\mathcal{R} \preceq \mathcal{M}$, in particular, M is a real closed field.

Base case ($n = 1$): Let $U \subseteq M$ be open and let $a \in U$. Since the topology on M is the order topology, there are $b_1, b_2 \in M$ and $\varepsilon > 0$ such that $a \in (b_1 + \varepsilon, b_2 - \varepsilon)$ and $(b_1 - \varepsilon, b_2 + \varepsilon) \subseteq U$. By our choice of b_1, b_2, and ε, for all $(x, y) \in (b_1 - \varepsilon, b_1 + \varepsilon) \times (b_2 - \varepsilon, b_2 + \varepsilon)$, we have $a \in (x, y) \subseteq U$.

Inductive step: Let $U \subseteq M^{n+1}$ be open and let $(a_1, \ldots, a_{n+1}) \in U$. We want to show that X_U has nonempty interior for this U. Let $\pi_n : M^{n+1} \to M^n$ be projection onto the first n coordinates and let $\pi^1 : M^{n+1} \to M$ be projection onto the last coordinate. Since projection is an open map, $\pi^1(U)$ and $\pi_n(U)$ are open. Let $X_n \subseteq M^{2n}$ be the set

$$\{(b_{11}, b_{12}, \ldots, b_{n1}, b_{n2}) \in M^{2n} : [b_{11} < a_1 < b_{12}] \wedge \ldots \wedge [b_{n1} < a_n < b_{n2}] \wedge (b_{11}, b_{12}) \times \ldots \times (b_{n1}, b_{n2}) \subseteq \pi_n(U)\}.$$

By our inductive assumption, X_n has nonempty interior. Let V be a nonempty open subset of X_n. By a similar argument as in the base case, there are $c_1, c_2 \in M$ and $\varepsilon > 0$ such that for all $(x, y) \in (c_1 - \varepsilon, c_1 + \varepsilon) \times (c_2 - \varepsilon, c_2 + \varepsilon)$, we have $a_{n+1} \in (x, y) \subseteq \pi^1(U)$. Therefore, $V \times (c_1 - \varepsilon, c_1 + \varepsilon) \times (c_2 - \varepsilon, c_2 + \varepsilon)$ is an open subset of X_U. $\square$

We are now ready to prove the main theorem of this subsection.

Theorem 5.11. *Every open set definable in* $(\overline{\mathbb{R}}, \Gamma, \Delta)$ *is definable in* $(\overline{\mathbb{R}}, \Delta)$.

Proof. For $n \geq 1$, let

$$D_n = \{(a_1, \ldots, a_n) \in M^n : \{a_1, \ldots, a_n\} \text{ is } \mathrm{dcl}^{\mathcal{M}}_{\mathcal{L}_{or}}\text{-independent over } G \cup A \cup C\}.$$

To prove the theorem, we will apply Corollary 3.1 in [4]. To do this, we must also check that for all $n \geq 1$,

1. D_n is dense in M^n;
2. for every $a \in D_n$ and every open set $U \subseteq M^n$, if $\text{tp}^{\mathcal{M}}(a|C)$ is realized in U, then $\text{tp}^{\mathcal{M}}(a|C)$ is realized in $U \cap D_n$;
3. for every $a, b \in D_n$, if b realizes $\text{tp}^{\mathcal{M}}(a|C)$, then b realizes $\text{tp}^{\mathcal{M}^*}(a|C)$.

(1): Let $U := (c_{11}, c_{12}) \times \ldots \times (c_{n1}, c_{n2})$ be a basic open set in M^n. We want to show that there exists $a \in M^n$ with $a \in U \cap D_n$.

Let $S = G \cup A \cup C$. We first find $a_1 \in M$ such that $a_1 \in (c_{11}, c_{12}) \setminus \text{dcl}^{\mathcal{M}}_{\mathcal{L}_{or}}(S)$. We use κ-saturation of $\mathcal{M}$ to show that there is $x \in M$ such that $x \in (c_{11}, c_{12}) \setminus \text{dcl}^{\mathcal{M}}_{\mathcal{L}_{or}}(S)$.

Let $f_1, \ldots, f_l$ be $\mathcal{L}_{or}(C)$-definable functions from M^{2n} to M. For each i, let $X_i := \{f_i(ga) : ga \in (GA)^n\}$. By Lemma 5.3, the set $M \setminus \bigcup_{i=1}^{l} X_i$ is dense in M. In particular, there is $y \in (c_{11}, c_{12})$ such that for all i and all tuples $ga \in (GA)^n$, $y \neq f_i(ga)$. By κ-saturation of $\mathcal{M}$, there is $a_1 \in M$ with $a_1 \in (c_{11}, c_{12}) \setminus \text{dcl}^{\mathcal{M}}_{\mathcal{L}_{or}}(S)$.

We now want to show that there is $a_2 \in M$ such that $(a_1, a_2) \in (c_{11}, c_{12}) \times (c_{21}, c_{22})$ and (a_1, a_2) is $\text{dcl}^{\mathcal{M}}_{\mathcal{L}_{or}}$-independent over S. By the exchange property of $\text{dcl}^{\mathcal{M}}_{\mathcal{L}_{or}}$, it is enough to find $a_2 \in (c_{21}, c_{22}) \setminus \text{dcl}^{\mathcal{M}}_{\mathcal{L}_{or}}(S \cup \{a_1\})$. But such an a_2 exists by a similar proof as in the previous paragraph. Continuing in this way, we can find $a_1, \ldots, a_n \in M$ such that $(a_1, \ldots, a_n) \in D_n \cap U$.

(2): Let $a \in D_n$ and $U \subseteq M^n$, and suppose $\text{tp}^{\mathcal{M}}(a|C)$ is realized in U. Let d be a realization of $\text{tp}^{\mathcal{M}}(a|C)$ in U. We will show that every finite subset of $\text{tp}^{\mathcal{M}}(a|C)$ is realized in $U \cap D_n$. By κ-saturation of $\mathcal{M}$, this suffices to prove that $\text{tp}^{\mathcal{M}}(a|C)$ is satisfied in $U \cap D_n$.

Let $\varphi_1(x), \ldots, \varphi_n(x) \in \text{tp}^{\mathcal{M}}(a|C)$ (so $\varphi_1, \ldots, \varphi_n$ are $\mathcal{L}(C)$-formulas). By Corollary 4.1.7 in Tychonievich [26], for each i, $\varphi_i(x)$ is equivalent to a formula of the form $\exists y \in V^{m_i} \theta_i(y, x)$. where θ_i is an $\mathcal{L}_{or}(C)$-formula. We claim that for each i, the set

$$A_i := \{x \in M^n : \mathcal{M} \models \exists y \in V^{m_i} \theta_i(y, x)\}$$

has interior, and its interior contains d.

Let $\text{int}(A_i)$ denote the interior of A_i. First note that since $\text{tp}^{\mathcal{M}}(a|C) = \text{tp}^{\mathcal{M}}(d|C)$, we have $d \in A_i$. Fix $\alpha \in A^{m_i}$ such that $\mathcal{M} \models \theta_i(\alpha, d)$, and let $B_i(\alpha) = \{x \in M^n : \mathcal{M} \models \theta_i(\alpha, x)\}$. Since $\mathcal{N}$ is o-minimal, let $\mathcal{D}$ be a decomposition of M^n into cells which partitions $B_i(\alpha)$, and let X be the cell in this decomposition which contains d. Let X be an $(i_1, \ldots, i_n)$-cell. We will show that X is an open cell. Suppose not; then for some $j \in \{1, \ldots, n\}$, we must have $i_j = 0$. As stated in Section 3.2, since $B_i(\alpha)$ is definable by an $\mathcal{L}_{or}(A \cup C)$-formula, X is also definable by an $\mathcal{L}_{or}(A \cup C)$-formula. Since $i_j = 0$, there is an $\mathcal{L}_{or}(C)$-definable function f and parameters $\beta \in A^l$ such that $f(\beta, d_1, \ldots, d_{j-1}) = d_j$. Consider the $\mathcal{L}(C)$-formula

$$\exists y_1 \in V \ldots \exists y_l \in V \; f(y_1, \ldots, y_l, x_1, \ldots, x_{j-1}) = x_j.$$

This formula is in $\mathrm{tp}^{\mathcal{M}}(d|C)$, hence it is also in $\mathrm{tp}^{\mathcal{M}}(a|C)$. But then there are parameters $\beta' \in A^l$ such that $f(\beta', a_1, \ldots, a_{j-1}) = a_j$. Therefore, $a_j \in \mathrm{dcl}^{\mathcal{M}}_{\mathcal{L}_{or}}(S)$, contradicting our assumption that $a \in D_n$. Therefore, X is an open cell containing d, so $d \in \mathrm{int}(A_i)$.

Now let $V = U \cap (\bigcap_{i=1}^n \mathrm{int}(A_i))$. Since V is a finite intersection of open sets, V is open. Moreover, since $d \in \mathrm{int}(A_i)$ for each i and since $d \in U$, V is nonempty. Since D_n is dense in M^n, there is $b \in M$ such that $b \in V \cap D_n$. Since $b \in \mathrm{int}(A_i) \subseteq A_i$ for each i, we have $\mathcal{M} \models \varphi_i(b)$ for each i. Therefore, $\mathrm{tp}^{\mathcal{M}}(a|C)$ is finitely satisfiable in $U \cap D_n$. By κ-saturation of $\mathcal{M}$, it is satisfiable in $U \cap D_n$.

(3): Let $a, b \in D_n$ and suppose that b satisfies $\mathrm{tp}^{\mathcal{M}}(a|C)$. We want to show that b satisfies $\mathrm{tp}^{\mathcal{M}^*}(a|C)$. Note that since $\mathcal{M}^* \succeq (\overline{\mathbb{R}}, \Gamma, \Delta)$, we have $\mathcal{M}^* \models T$. Let $\mathcal{I}$ denote the back-and-forth system constructed in Theorem 5.6 for $\mathcal{M}^*$ and $\mathcal{M}^*$. To show that b satisfies $\mathrm{tp}^{\mathcal{M}^*}(a|C)$, it suffices to show that there is $\iota \in \mathcal{I}$ such that ι fixes C pointwise and $\iota(a_i) = b_i$ for all i. Let $\mu : \{a_1, \ldots, a_n\} \to \{b_1, \ldots, b_n\}$ be the function defined by $\mu(a_i) = b_i$ for each i.

Note that since $a, b \in D_n$, $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are both algebraically independent over $\mathbb{Q}(GA)$. We first construct $A^\infty \subseteq A$ as in Lemma 5.10. Let $p = \mathrm{tp}^{\mathcal{M}^*}_{\mathcal{L}_{or}}(A^\infty|a)$. By our assumption that $\mathrm{tp}^{\mathcal{M}}(a) = \mathrm{tp}^{\mathcal{M}}(b)$, p is finitely satisfiable by elements of A in $\mathcal{M}^*$. Thus, by κ-saturation of $\mathcal{M}^*$, there is a subset B^∞ of elements of A such that $\mu(\mathrm{tp}^{\mathcal{M}^*}_{\mathcal{L}_{or}}(A^\infty|a)) \subseteq \mathrm{tp}^{\mathcal{M}^*}_{\mathcal{L}_{or}}(B^\infty|b)$.

Let $F = \mathbb{Q}(\mathrm{Re}(\Gamma\Delta))$ and let

$$K' = F(a, A^\infty)^{\mathrm{rc}}, \quad G' = \langle \Gamma \rangle_G, \quad A' = A^\infty,$$
$$L' = F(b, B^\infty)^{\mathrm{rc}}, \quad H' = \langle \Gamma \rangle_G, \quad B' = B^\infty.$$

By construction, $\mu(\mathrm{tp}^{\mathcal{M}^*}_{\mathcal{L}_{or}}(A^\infty|a)) \subseteq \mathrm{tp}^{\mathcal{M}^*}_{\mathcal{L}_{or}}(B^\infty|b)$. So we have an ordered field isomorphism $\iota : K' \to L'$ which takes A^∞ to B^∞ and a to b. The fact that $\iota \in \mathcal{I}$ follows as in Lemma 5.10.

To finish the proof, we must show that if $U \subseteq \mathbb{R}^m$ is an open definable set in $\mathcal{R}^*$, then U is definable in $\mathcal{R}$. Let $U \subseteq \mathbb{R}^m$ be an open definable set in $\mathcal{R}^*$. Then U is definable with finitely many parameters from $\mathbb{R}$, say $\{\alpha_1, \ldots, \alpha_n\}$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and let $\phi(y, x)$ be an $\mathcal{L}'$-formula defining U, so that

$$U = \{x \in \mathbb{R}^m : \mathcal{R}^* \models \phi(\alpha, x)\}.$$

Since $\mathcal{R}^* \preceq \mathcal{M}^*$, the set $V := \{x \in M^m : \mathcal{M}^* \models \phi(\alpha, x)\}$ is open and definable (in $\mathcal{M}^*$) over the set $\{\alpha_1, \ldots, \alpha_n\}$. By Corollary 3.1 in [4], V is definable in $\mathcal{M}$ over $\{\alpha_1, \ldots, \alpha_n\}$. Let ψ be an L-formula such that $V = \{x \in M^m : \mathcal{M} \models \psi(\alpha, x)\}$. Now consider the definable set $U' := \{x \in \mathbb{R}^m : \mathcal{R} \models \psi(\alpha, x)\}$. Since $\mathcal{R} \preceq \mathcal{M}$ and $\mathcal{R}^* \preceq \mathcal{M}^*$, we have $U' = U$. Therefore, U is definable in $\mathcal{R}$. $\square$

From this theorem, Theorem B follows immediately.

Corollary 5.12. *The open core of* $(\overline{\mathbb{R}}, a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}})$ *is interdefinable with* $(\overline{\mathbb{R}}, a^{\mathbb{Z}})$.

Chapter 6

Subgroups of $\mathbb{C}^\times$ generated by a complex number and a positive real number

We next consider expansions of $\overline{\mathbb{R}}$ by a subgroup of $\mathbb{C}$ generated by a complex number and a positive real number. Such subgroups have the form $(ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}}$, where $a, b > 0$ and $\varphi \in \mathbb{R}$. In contrast to the subgroups studied in Chapter 5, these groups vary greatly in appearance. We have already seen two possibilities: the group might consist of concentric rings in $\mathbb{C}$ (as in Fig. 5.1) or might look like a star (as in Fig. 4.2). The group can also lie in a logarithmic spiral (as discussed after Corollary 4.11) or be dense in $\mathbb{C}$.

We first concentrate on the case where the group $(ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}}$ is dense in $\mathbb{C}$ and prove Theorem C under this assumption. In Section 6.4, we consider expansions of the form $(\overline{\mathbb{R}}, (ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}})$, where $(ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}}$ is not necessarily dense in $\mathbb{C}$.

6.1 Axiomatizing expansions by dense groups

Let $a, b \in \mathbb{R}^{>0}$ with $a, b > 1$. In the rest of this chapter, let $\Xi = a^{\mathbb{Z}}b^{\mathbb{Z}}$ and let $\Delta = b^{\mathbb{Z}}$. Let $\Gamma = (e^{i\varphi})^{\mathbb{Z}}$, where $\varphi \in \mathbb{R}$. Let $\rho : \Xi \to \Gamma$ be given by $\rho(a^k b^l) = e^{i\varphi k}$ for $k, l \in \mathbb{Z}$.

Proposition 6.1. *Let $H = (ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}}$, where $\frac{\ln(a)}{\ln(b)} \notin \mathbb{Q}$. $(\overline{\mathbb{R}}, H)$ and $(\overline{\mathbb{R}}, b^{\mathbb{Z}}, a^{\mathbb{Z}}b^{\mathbb{Z}}, (e^{i\varphi})^{\mathbb{Z}}, \rho)$ are interdefinable.*

Proof. We have

$$H = \{x\rho(x) : x \in a^{\mathbb{Z}}b^{\mathbb{Z}}\}.$$

So H is definable in $(\overline{\mathbb{R}}, b^{\mathbb{Z}}, a^{\mathbb{Z}}b^{\mathbb{Z}}, (e^{i\varphi})^{\mathbb{Z}}, \rho)$.

Conversely, we have

$$b^{\mathbb{Z}} = \{z \in H : \mathrm{Im}(z) = 0\}$$

$$a^{\mathbb{Z}}b^{\mathbb{Z}} = \{x \in \mathbb{R} : \exists z \in H(|z| = x)\}.$$

$$(e^{i\varphi})^{\mathbb{Z}} = \{z \in \mathbb{C} : \exists w \in H(z = \pi(w))\}.$$

As stated after Definition 4.7, ρ is definable in $(\overline{\mathbb{R}}, H)$. So $b^{\mathbb{Z}}, a^{\mathbb{Z}}b^{\mathbb{Z}}, (e^{i\varphi})^{\mathbb{Z}}$, and ρ are definable in $(\overline{\mathbb{R}}, H)$. $\square$

In this section, we will assume that the graph of ρ is dense in $\mathbb{R}^{>0} \times \mathbb{S}^1$. As proved in Corollary 4.11, this is equivalent to $(ae^{i\varphi})^{\mathbb{Z}} b^{\mathbb{Z}}$ being dense in $\mathbb{C}$. In particular, $a^{\mathbb{Z}} b^{\mathbb{Z}}$ is dense in $\mathbb{R}^{>0}$ and $(e^{i\varphi})^{\mathbb{Z}}$ is dense in $\mathbb{S}^1$.

As before, the first step in proving Theorem C is to obtain an axiomatization for $(\overline{\mathbb{R}}, b^{\mathbb{Z}}, a^{\mathbb{Z}} b^{\mathbb{Z}}, (e^{i\varphi})^{\mathbb{Z}}, \rho)$ when we add constants for each element of $a^{\mathbb{Z}}$, $a^{\mathbb{Z}} b^{\mathbb{Z}}$, and $(e^{i\varphi})^{\mathbb{Z}}$. As in Section 5.1, we begin by defining a theory T_2 whose models behave similarly to $(\overline{\mathbb{R}}, b^{\mathbb{Z}}, a^{\mathbb{Z}} b^{\mathbb{Z}}, (e^{i\varphi})^{\mathbb{Z}}, \rho)$

Let U and V be unary predicates and let P be a binary predicate. Let μ_1, μ_2 be unary function symbols. For the rest of this chapter, let $\mathcal{L}$ denote the language $\mathcal{L}_{or}(U, V, P, \mu_1, \mu_2)$. For convenience, we will usually write $\mathcal{L}$-structures in the form (K, A, B, G, μ). We write $\mathcal{L}(\Xi, \Gamma)$ for the language $\mathcal{L}$ together with constants for each element of Ξ and of $\mathrm{Re}(\Gamma) \cup \mathrm{Im}(\Gamma)$.

Suppose $(K, A, B, G, \mu_1, \mu_2)$ is an $\mathcal{L}$-structure, where K is a real closed field. In such a structure, we interpret μ_1 and μ_2 as the real and imaginary parts of a function $\mu : K \to K(i)$. Thus, we will also write $\mu(x)$ instead of $\mu_1(x) + i\mu_2(x)$.

Definition 6.2. Let T_2 be the $\mathcal{L}(\Xi, \Gamma)$-theory whose models have the form

$$(K, A, B, G, \mu, (\xi')_{\xi \in \Xi}, (\gamma')_{\gamma \in \Gamma})$$

such that:

1. K is a real closed field
2. $\delta \mapsto \delta' : \Delta \to A$, $\xi \mapsto \xi' : \Xi \to B$, and $\gamma \mapsto \gamma' : \Gamma \to G$ are group homomorphisms
3. if $\xi' \in A$, then $\xi \in \Delta$
4. B is a subgroup of $K^{>0}$ that is dense in $K^{>0}$
5. A is a subgroup of $K^{>0}$ with a smallest element ε' greater than 1 such that $A \subseteq B$
6. for all $x \in K^{>0}$, there is $y \in A$ such that $y \leq x < y\varepsilon'$
7. the graph of μ is dense in $K^{>0} \times \mathbb{S}^1(K)$
8. the restriction of μ to B is a surjective group homomorphism onto G and if $b \in K \setminus B$, then $\mu(b) = 1$
9. $\ker(\mu) \cap B = A$
10. for all $\xi \in \Xi$, $\mu(\xi) = (\rho(\xi))'$
11. for all $b \in B$ and $n > 0$, if $b^n = \xi'$, then $b \in \Xi$.

12. for all $g \in G$ and $n > 0$, if $g^n = \gamma'$, then $g \in \Gamma$.

13. $G_{\text{tor}} = \Gamma_{\text{tor}}$

14. for every $n > 0$, there are $k_n \geq 1$ and $\xi_{n1}, \ldots, \xi_{nk_n} \in \Xi$ such that for all $b \in B$, b is equivalent to one of $\xi_{n1}, \ldots, \xi_{nk_n}$ modulo $B^{[n]}$

15. $(K, (\gamma\xi)_{\gamma\in\Gamma,\xi\in\Xi})$ satisfies the orientation axioms for $\Gamma\Xi$

16. $(K, GB, (\gamma\xi))$ satisfies the Mann axioms for $\Gamma\Xi$

Clearly, $(\overline{\mathbb{R}}, b^{\mathbb{Z}}, a^{\mathbb{Z}}b^{\mathbb{Z}}, (e^{i\varphi})^{\mathbb{Z}}, \rho)$ satisfies all of the axioms in T_2.

Since $\Gamma\Delta \subseteq \Gamma\Xi$, it is clear from definition of the Mann property that $\Gamma\Delta$ has the Mann property.

Theorem 6.3. *If $\mathcal{M} \models T$, then $(K, GA, (\gamma\delta))$ satisfies the Mann axioms for $\Gamma\Delta$.*

Proof. Since $\mathcal{M} \models T$, $(K, GB, (\gamma\xi))$ satisfies the Mann axioms for $\Gamma\Xi$. Let $a_1, \ldots, a_n \in \mathbb{Q}$ and let $\gamma_1\delta_1, \ldots, \gamma_m\delta_m \in (\Gamma\Delta)^n$ be the solutions to the equation

$$a_1x_1 + \ldots + a_nx_n = 1$$

in $\Gamma\Delta$. Let $\eta_1\xi_1, \ldots, \eta_m\xi_m$ be the solutions to this equation in $\Gamma\Xi$.

Suppose we have $(y_1, z_1), \ldots, (y_n, z_n) \in GA$ such that

$$a_1(y_1 + iz_1) + \ldots + a_n(y_n + iz_n) = 1.$$

Let $y = (y_1, \ldots, y_n)$ and let $z = (z_1, \ldots, z_n)$. We want to show that for some $j \in \{1, \ldots, n\}$, $(y, z) = \gamma_j\delta_j$ in the sense defined in Section 3.1 of [5]. Since $(K, GB, (\gamma\xi))$ satisfies the Mann axioms for $\Gamma\Xi$, there is $i \in \{1, \ldots, m\}$ such that $(y, z) = \eta_i\xi_i$. Fix $l \in \{1, \ldots, n\}$. Since $(y_l, z_l) \in GA$, there are $g_l \in G$ and $a_l \in A$ such that $(y_l, z_l) = g_la_l$. We have $(y_l, z_l) = \eta_{il}\xi_{il} = g_la_l$, so since $G \cap B = \{1\}$, we have $a_l = \xi_{il}$ and $g_l = \eta_{il}$. By (3) in our theory T_2, we have $a_l \in \Delta$. This proves that $\eta_i\xi_i \in (\Gamma\Delta)^n$. Since $\Gamma\Delta$ has the Mann property, we must have $\eta_i\xi_i = \gamma_j\delta_j$ for some j. $\square$

6.1.1 Substructures of models of T_2

In this section, we define a collection of $\mathcal{L}$-structures in analogy with Definition 5.4.

Let $\mathcal{M} := (K, A, B, G, \mu, (\xi)_{\xi\in\Xi}, (\gamma)_{\gamma\in\Gamma})$ be a model of T_2. Let κ be an infinite cardinal with $\kappa > |\Gamma\Xi|$.

Definition 6.4. Let $\mathcal{S}(\mathcal{M})$ be the collection of $\mathcal{L}$-structures (K', A', B', G', μ') such that:

1. K' is a real closed subfield of K of cardinality less than κ
2. G' is a pure subgroup of G containing Γ
3. B' is a pure subgroup of B containing Ξ
4. $A' = B' \cap A$
5. $K'(i)$ and $\mathbb{Q}(GB)$ are free over $\mathbb{Q}(G'B')$
6. $\mu' = \mu|_{K'}$ and $\mu'(B') = G'$
7. for all $k \in (K')^{>0}$, there is $a \in A'$ such that $a \leq k < a\varepsilon$.

Next we make some observations about $\mathcal{L}$-structures (K', A', B', G', μ') that are elements of $\mathcal{S}(\mathcal{M})$.

Lemma 6.5. *Let $(K', A', B', G', \mu') \in \mathcal{S}(\mathcal{M})$. Then:*

1. *$G'A' = G'B' \cap GA$;*
2. *A' is a pure subgroup of A;*
3. *$\Delta \subseteq A'$, and*
4. *$\mu'(K') = G'$.*

Proof. (1): Since $G \cap B = \{1\}$, by condition (4) in Definition 6.4, $G'A' = G'B' \cap GA$.

(2): By conditions (3) and (4) in Definition 6.4, A' is a pure subgroup of A. To see this, let $a' \in A'$ be such that $a' = a^n$ for some $a \in A$. Since $A' = B' \cap A$, we have $a' \in B'$. By purity of B' in B, there is $x \in B'$ such that $a' = x^n = a^n$. Since K is real closed, $x = a$, so $a \in B' \cap A$, so $a \in A'$.

(3): For all $\delta \in \Delta$, we have $\delta \in B' \cap A$ since (by condition (3) in Definition 6.4), $\Xi \subseteq B'$. So $\Delta \subseteq A'$.

(4): If $k \in K' \setminus B$, then since $\mathcal{M} \models T$, $\mu'(k) = \mu(k) = 1$. Therefore, we also have $\mu'(K') = G'$. $\square$

The next lemma is the analogue of Lemma 6.0.7 in [17]. We will need this lemma to prove that $\mathcal{L}$-structures in $\mathcal{S}(\mathcal{M})$ are actually substructures of $\mathcal{M}$.

Lemma 6.6. *Let K be a field with subgroups A, B of $K^{>0}$ such that $A \subseteq B$, and let G, G' be subgroups of $\mathbb{S}^1(K)$ with $G' \subseteq G$. Let K' be a subfield of K with subgroups A', B' of $(K')^{>0}$ such that $B' \subseteq B$ and $G'A' = G'B' \cap GA$.*

Suppose that for all $q_1, \ldots, q_n \in \mathbb{Q}^\times$ the equation $q_1x_1 + \ldots + q_nx_n = 1$ has the same nondegenerate solutions in $G'B'$ as in GB, and that $K'(i)$ and $\mathbb{Q}(GB)$ are free over $\mathbb{Q}(G'B')$. Then $K'(i)$ and $\mathbb{Q}(GA)$ are free over $\mathbb{Q}(G'A')$.

Proof. By Fact 2.5, to show that $K'(i)$ and $\mathbb{Q}(GA)$ are free over $\mathbb{Q}(G'A')$, it suffices to show that if $B \subseteq GA$ is a transcendence basis of $\mathbb{Q}(GA)$ over $\mathbb{Q}(G'A')$, then B is algebraically independent over $K'(i)$.

A similar proof as in Lemma 6.0.7 in [17] shows that for any $z_1, \ldots, z_m \in GA$ which are algebraically dependent over $K'(i)$, $z_1, \ldots, z_m$ are algebraically dependent over $\mathbb{Q}(G'A')$. Let $B \subseteq GA$ be a transcendence basis of $\mathbb{Q}(GA)$ over $\mathbb{Q}(G'A')$. In particular, B is algebraically independent over $\mathbb{Q}(G'A')$. Then B is algebraically independent over $K'(i)$, as desired.

□

Lemma 6.7. *If $\mathcal{M}' := (K', A', B', G', \mu')$ satisfies conditions (1)-(6) in Definition 6.4, then $\mathcal{M}'$ is a substructure of $\mathcal{M}|_{\mathcal{L}}$. If $\mathcal{M}'$ is a substructure of $\mathcal{M}|_{\mathcal{L}}$, then satisfying condition (7) in Definition 6.4 is equivalent to K' being closed under λ.*

Proof. We must show that $G \cap K'(i) = G'$, $B \cap K' = B'$, $A \cap K' = A'$, $\mu' = \mu|_{K'}$, and $\mu'(K') \subseteq K'(i)$.

The proof that $G \cap K'(i) = G'$ is the same as in Lemma 5.5. Likewise, the proof that $B \cap K' = B'$ is the same as the proof that $A \cap K' = A'$ in the proof of Lemma 5.5. Note that the proof that $B \cap K' = B'$ only uses conditions (3) and (4) in Definition 6.4. By Lemma 6.6, $K'(i)$ and $\mathbb{Q}(GA)$ are free over $\mathbb{Q}(G'A')$. As proved above, A' is pure in A and $(K, GA, (\gamma\delta))$ satisfies the Mann axioms for $\Gamma\Delta$. Therefore, $A \cap K' = A'$.

By condition (6) in Definition 6.4, $\mu' = \mu|_{K'}$ and $\mu'(K') \subseteq K'(i)$.

By a similar proof as in Lemma 5.5, if $\mathcal{M}'$ is a substructure of $\mathcal{M}$, then K' is closed under λ. □

6.1.2 Elementary equivalence of models of T_2

In this section, we give conditions for models of T_2 to be elementarily equivalent. The next theorem is the analogue of Theorem 5.6.

Theorem 6.8. *Two models*

$$\mathcal{M} := (K, A, B, G, \mu, (\xi)_{\xi \in \Xi}, (\gamma)_{\gamma \in \Gamma})$$

and

$$\mathcal{N} := (L, C, D, H, \nu, (\xi)_{\xi \in \Xi}, (\gamma)_{\gamma \in \Gamma})$$

of T_2 are elementarily equivalent if and only if $[n]G = [n]H$ for $n \geq 1$, and for all $\gamma \in \Gamma$, $\xi \in \Xi$, and $n \geq 1$,

$$\gamma \textit{ is an } n\textit{th power in } G \textit{ if and only if } \gamma \textit{ is an } n\textit{th power in } H, \textit{ and}$$

$$\xi \textit{ is an } n\textit{th power in } B \textit{ if and only if } \xi \textit{ is an } n\textit{th power in } D.$$

We may assume that $\mathcal{M}$ and $\mathcal{N}$ are κ-saturated for some $\kappa > |\Gamma\Xi|$.

The "only if" direction of the theorem is clear. Let $\mathcal{I}$ be the set of isomorphisms between members of $\mathcal{S}(\mathcal{M})$ and $\mathcal{S}(\mathcal{N})$ that fix Γ and Ξ pointwise. To prove the "if" direction of the theorem, we will show that $\mathcal{I}$ is a nonempty back-and-forth system.

$\mathcal{I}$ is nonempty

Let

$$K' = \mathbb{Q}(\mathrm{Re}(\Gamma\Xi))^{\mathrm{rc}}, A' = \Delta, B' = \{b \in B : b^n \in \Xi \text{ for some } n > 0\},$$

$$G' = \{g \in G : g^n \in \Gamma \text{ for some } n > 0\}, \mu' = \mu|_{K'}$$

and let

$$L' = \mathbb{Q}(\mathrm{Re}(\Gamma\Xi))^{\mathrm{rc}}, C' = \Delta, D' = \{d \in D : d^n \in \Xi \text{ for some } n > 0\},$$

$$H' = \{h \in H : h^n \in \Gamma \text{ for some } n > 0\}, \nu' = \nu|_{L'}.$$

By axioms (11) and (12) in T_2, $B' = D' = \Xi$ and $G' = H' = \Gamma$.

Let $\mathcal{M}' := (K', A', B', G', \mu')$ and let $\mathcal{N}' := (L', C', D', H', \nu')$.

Clearly K' is a real closed subfield of K. By a similar proof as in Section 5.1.2, $G' \subseteq (K')^2$ and $A' \subseteq K'$. By definition of B', $B' \subseteq K'$ and B' is a pure subgroup of B.

Next we check that $A' = B' \cap A$. Since $\Delta \subseteq \Xi$, it is clear that $A' \subseteq B' \cap A$. Now let $a \in B' \cap A$. By definition of B', there are $n \in \mathbb{N}$ and $\xi \in \Xi$ such that $a^n = \xi$. Therefore, $\xi \in A$, and by axiom (3) of T_2, $\xi \in \Delta$. By Lemma 2.18, Δ is a pure subgroup of A, so $a \in A'$.

The fact that $K'(i)$ and $\mathbb{Q}(GB)$ are free over $\mathbb{Q}(G'B')$ follows from Corollary 2.8 with $X = \emptyset$ and $k = \mathbb{Q}(\mathrm{Re}(\Gamma\Xi))$.

By definition, $\mu' = \mu|_{K'}$. Since (K', A', B', G', μ') satisfies conditions (3) and (4) of Definition 6.4, we have $B' = K' \cap B$. Next we check that $\mu'(B') = G'$. By axiom (10) in T_2, $\mu'|_{\Xi} = \rho$, so we have $\mu'(\Xi) = \rho(\Xi) = \Gamma$.

We have now proved that $\mathcal{M}'$ satisfies conditions (1)-(6) in Definition 6.4. Since we assume that $(K, (\gamma\xi)_{\gamma\in\Gamma,\xi\in\Xi})$ satisfies the orientation axioms for $\Gamma\Xi$, every positive element of K' is finite. Therefore, by Lemma 2.17, we have $\lambda((K')^{>0}) = \Delta$. This proves that K' is closed under λ.

Let $\iota : K' \to L'$ be the natural function extending the identity on $\mathrm{Re}(\Gamma\Xi)$. We will show that $\iota \in \mathcal{I}$. Since we assume that $\mathcal{M}$ and $\mathcal{N}$ satisfy the orientation axioms for $\Gamma\Xi$, ι is an ordered field isomorphism. Clearly $\iota(A') = C'$. By assumption, for each $\xi \in \Xi$ and $n > 0$, ξ is an nth power in B if and only if ξ is an

nth power in D. Therefore, $\iota(B') = D'$. We also assume that for each $\gamma \in \Gamma$ and $n > 0$, γ is an nth power in G if and only if γ is an nth power in H. Since $H_{\mathrm{tor}} = \Gamma_{\mathrm{tor}}$, $\iota(G') = H'$.

To show that ι is an isomorphism of $\mathcal{L}$-structures, it remains to prove that $\iota(\mu'(k)) = \nu'(\iota(k))$ for all $k \in K'$. Since μ, ν both extend ρ, for any $\xi \in \Xi$, $\iota(\mu(\xi)) = \nu(\xi)$. If $k \in B$, then $k \in K' \cap B = B' = \Xi$. Therefore, we again have $\iota(\mu(\xi)) = \nu(\xi)$.

Now suppose that $k \notin B$. Since ι takes B' to D', we also have $\iota(k) \notin D$. By axiom (8) in T_2, we have $\mu(k) = \nu(\iota(k)) = 1$. Therefore, $\iota(\mu(k)) = 1 = \nu(\iota(k))$.

Therefore, ι is an isomorphism between the $\mathcal{L}$-structures $\mathcal{M}'$ and $\mathcal{N}'$ which is the identity on $\Gamma\Xi$. This proves that $\iota \in \mathcal{I}$.

$\mathcal{I}$ is a back-and-forth system

Let $\mathcal{M}' := (K', A', B', G', \mu') \in \mathcal{S}(\mathcal{M})$ and $\mathcal{N}' := (L', C', D', H', \nu') \in \mathcal{S}(\mathcal{N})$. Let $\iota : \mathcal{M}' \to \mathcal{N}'$ be in $\mathcal{I}$, and let $a \in K \setminus K'$.

We want to extend ι to $\iota' \in \mathcal{I}$ such that a is in the domain of ι'. We must also show that if $a \in L \setminus L'$ and $\iota \in \mathcal{I}$, then ι can be extended to $\iota' \in \mathcal{I}$ with a in the range of ι'. However, as in the proof of Theorem 5.6, the proof of this will be the same as the proof of the "forth" case.

Since we now have a function μ in our language, when extending an element ι in our back-and-forth system to include x in its domain, we must also consider where to map $\mu(x)$. This is the main difference between this proof and the proof of Theorem 5.6.

We have four cases:

1. $a \in A$
2. $a \in B$
3. $a \in K'(\mathrm{Re}(GB))^{\mathrm{rc}}$
4. $a \in K \setminus K'(\mathrm{Re}(GB))^{rc}$

Case 1. $a \in A$.

Define sets $\Sigma_1, \Sigma_2, \Sigma_3$ of $\mathcal{L}(K')$-formulas in the variable x by

$$\Sigma_1 := \{\iota(k_1) < x < \iota(k_2) : k_1 < a < k_2, k_1, k_2 \in K'\}$$

$$\Sigma_2 := \{\iota(a')x^l \in C^{[m]} : a' \in A', l \in \mathbb{Z}, m > 0, a'a^l \in A^{[m]}\}$$

$$\Sigma_3 := \{\iota(b')x^k \in D^{[n]} : b' \in B', k \in \mathbb{Z}, n > 0, b'a^k \in B^{[n]}\}.$$

We will find $c \in C$ such that c satisfies $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3$.

Since A and C are regularly discrete abelian groups, we have $[p]A = [p]C = p$ for each prime p. By axiom (14) in T_2, we have $[p]B = [p]D$ for each prime p as well. Therefore, we may apply Lemma 4.2.1 in [17]. Using this lemma, we fix $h \in C$ such that for all $a' \in A'$, $b' \in B'$, $m, n > 0$, and $l, k \in \mathbb{Z}$,

$$a'a^l \in A^{[m]} \Leftrightarrow \iota(a')h^l \in C^{[m]}$$

and

$$b'a^k \in B^{[m]} \Leftrightarrow \iota(b')h^l \in D^{[m]}.$$

By κ-saturation of $\mathcal{N}$, to show that $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ is satisfied by an element of C, it suffices to show that $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ is finitely satisfiable in C. To prove this, fix $k_1, k_2 \in K'$ such that $k_1 < a < k_2$. We will show that there exists $\beta \in C$ such that $\iota(k_1) < \beta < \iota(k_2)$ and β satisfies $\Sigma_2 \cup \Sigma_3$. Using κ-saturation and the regular discreteness of C, we can find $\eta \in (\iota(k_1)h^{-1}, \iota(k_2)h_1^{-1}) \cap \bigcap_{k \geq 1} C^{[k]}$. (The proof of this is similar to the proof of Case 1 in Theorem 5.6.) Let $\beta = h\eta$. By our choice of β, we have $\iota(k_1) < \beta < \iota(k_2)$ and β satisfies $\Sigma_2 \cup \Sigma_3$.

By κ-saturation of $\mathcal{N}$, we obtain $c \in C$ such that $\mathcal{N} \models \Sigma_1(c) \cup \Sigma_2(c) \cup \Sigma_3(c)$. Since c satisfies the same cut over K' that a does over L', we have an $\mathcal{L}_{or}$-isomorphism $\iota' : K'(a)^{\mathrm{rc}} \to L'(c)^{\mathrm{rc}}$ extending ι which takes a to c. Since c satisfies Σ_2, we have $\iota'(A'\langle a\rangle_A) = C'\langle c\rangle_C$. Likewise, since c satisfies Σ_3, we have $\iota'(B'\langle a\rangle_B) = D'\langle c\rangle_D$.

Let $\mu'' : K'(a)^{\mathrm{rc}} \to G$ be defined by $\mu'' = \mu|_{K'(a)^{\mathrm{rc}}}$. Similarly, let $\nu'' : L'(c)^{\mathrm{rc}} \to H$ be defined by $\nu'' = \nu|_{L'(c)^{\mathrm{rc}}}$. Now let

$$\mathcal{M}'' := (K'(a)^{\mathrm{rc}}, A'\langle a\rangle_A, B'\langle a\rangle_B, G', \mu'')$$

$$\mathcal{N}'' := (L'(c)^{\mathrm{rc}}, C'\langle c\rangle_C, D'\langle c\rangle_D, H', \nu'').$$

We will show that $\mathcal{M}'' \in \mathcal{S}(\mathcal{M})$. (The proof that $\mathcal{N}'' \in \mathcal{S}(\mathcal{N})$ is similar.) It is easy to check that the first 3 conditions in Definition 6.4 are satisfied for $\mathcal{M}''$. Next we will show that $A'\langle a\rangle_A = B'\langle a\rangle_B \cap A$. Let $x \in B'\langle a\rangle_A \cap A$. Then there are $a' \in A', b \in B', l \in \mathbb{Z}$, and $m > 0$ such that $x = (ba^l)^{1/m} = a'$. Then $b = (a')^m a^{-l}$, so $b \in A \cap B'$. By our assumption that $\mathcal{M}' \in \mathcal{S}(\mathcal{M})$, we have $A \cap B' = A'$. Therefore, $b \in A'$. By definition of $A'\langle a\rangle_A$, we have $x \in A'\langle a\rangle_A$. It is clear that $A'\langle a\rangle_A \subseteq B'\langle a\rangle_A \cap A$, so $A'\langle a\rangle_A = B'\langle a\rangle_A \cap A$.

Next we will show that $K'(a)^{\mathrm{rc}}(i)$ and $\mathbb{Q}(GB)$ are free over $\mathbb{Q}(G'B'\langle a\rangle_B)$. By assumption, $K'(i)$ and

$\mathbb{Q}(GB)$ are free over $\mathbb{Q}(G'B')$. Therefore, we may apply Lemma 6.6 with $E = \{a\}$ and $X = \emptyset$ to get that $K'(a)^{\mathrm{rc}}(i)$ and $\mathbb{Q}(GB)$ are free over $\mathbb{Q}(G'B', a)$. Since $\mathbb{Q}(G'B', a) \subseteq \mathbb{Q}(G'B'\langle a\rangle_B)$, $K'(i)$ and $\mathbb{Q}(GB)$ are free over $\mathbb{Q}(G'B'\langle a\rangle_B)$.

By definition of μ'', we have $\mu'' = \mu|_{K'}$. Next we will show that $\mu''(B'\langle a\rangle_B) = G'$. Let $x \in B'\langle a\rangle_B$. By definition, there are $b \in B'$, $l \in \mathbb{Z}$, and $m > 0$ such that $x = (ba^l)^{1/m}$. Note that since $\mathcal{M} \models T$, we have $a^l \in \ker(\mu)$. Since μ is a group homomorphism, we have

$$(\mu''(x))^m = \mu(x^m) = \mu(b)\mu(a^l) = \mu(b) = \mu'(b) \in G'.$$

Since $\mu''(x) = \mu(x)$, we have $\mu''(x) \in G$. By purity of G' in G, there is $z \in G'$ such that $\mu'(b) = z^m$. Since $G_{\mathrm{tor}} = \Gamma_{\mathrm{tor}}$, there is $\gamma \in \Gamma$ such that $\mu''(x) = \gamma z$. Since $\Gamma \subseteq G'$, we have $\mu''(x) \in G'$, as desired.

Conversely, let $g \in G'$. Since $\mu'(B') = G'$ and $\mu''|_{B'} = \mu'|_{B'}$, there is $b \in B'$ such that $\mu''(b) = g$. We also have $b \in B'\langle a\rangle_B$, so $G' \subseteq \mu''(B'\langle a\rangle_B)$.

The proof that $K'(a)^{\mathrm{rc}}$ is closed under λ is the same as the proof of this fact in Case 1 of Theorem 5.6.

Next we will show that $\iota' \in \mathcal{I}$. To show this, it remains to prove that $\iota'(\mu''(k)) = \nu''(\iota'(k))$ for all $k \in K'(a)^{\mathrm{rc}}$.

First suppose $k \in B$. Since we have proved that $\mathcal{M}''$ satisfies (3) and (4) in Definition 6.4, we have $K'(a)^{\mathrm{rc}} \cap B = B'\langle a\rangle_B$. Thus, there are $b \in B'$, $n \in \mathbb{Z}$, and $m > 0$ such that $k = (ba^n)^{1/m}$. By Lemma 2.10, since A is regularly discrete with smallest positive element $\varepsilon \in \Delta$, there is $\delta \in \Delta$ such that $\delta a^n \in A^{[m]}$. Since we assume that $\iota \in \mathcal{S}$, we have $\Delta \subseteq A'$. Let $\alpha \in A$ be such that $\delta a^n = \alpha^m$. By axiom (9) in T_2, $A \subseteq \ker(\mu)$, so $\mu(\alpha) = 1$. Therefore, we have

$$\begin{aligned}
\iota'(\mu((ba^n)^{1/m})) &= \iota'(\mu((b\delta^{-1})^{1/m}(\delta a^n)^{1/m})) \\
&= \iota'(\mu((b\delta^{-1})^{1/m}))\iota'(\mu(\alpha)) \\
&= \iota'(\mu((b\delta^{-1})^{1/m}))
\end{aligned}$$

Now note that $(b\delta^{-1})^{1/m} = \dfrac{(ba^n)^{1/m}}{(\delta a^n)^{1/m}}$, so $(b\delta^{-1})^{1/m} \in B'\langle a\rangle_B$. In particular, $(b\delta^{-1})^{1/m} \in B$. Since $b\delta^{-1} \in K'$ and K' is real closed, we have $(b\delta^{-1})^{1/m} \in K'$. But $K' \cap B = B'$, so $(b\delta^{-1})^{1/m} \in B'$. By our assumption that $\iota : \mathcal{M}' \to \mathcal{N}'$ is an isomorphism, for any $k \in K'$, $\iota(\mu(k)) = \nu(\iota(k))$. Moreover, by construction, $\iota'|_{K'} = \iota$. Since $\alpha \in A'\langle a\rangle_A$, $\iota'(\alpha) \in C'\langle c\rangle_C$, and so $\nu(\iota'(\alpha)) = 1$. Since ι' is an $\mathcal{L}_{or}$-

isomorphism,

$$\begin{aligned}
\iota'(\mu((b\delta^{-1})^{1/m})) &= \nu(\iota'((b\delta^{-1})^{1/m})) \\
&= \nu(\iota'((b\delta^{-1})^{1/m}))\nu(\iota'(\alpha)) \\
&= \nu(\iota'((b\delta^{-1}\delta a^n)^{1/m})) \\
&= \nu(\iota'((ba^n)^{1/m})
\end{aligned}$$

Therefore, for any $k \in K'(a)^{\mathrm{rc}} \cap B$, we have $\iota'(\mu''(k)) = \nu''(\iota'(k))$.

If $k \in K'(a)^{\mathrm{rc}} \setminus B$, then $\mu''(k) = \mu(k) = 1$, so $\mu''(k) \in G'$. Since ι' takes $B'\langle a\rangle_B$ to $D'\langle c\rangle_D$, we have $\iota'(k) \notin D$. Therefore, we have

$$\iota'(\mu''(k)) = 1 = \nu''(\iota'(k)).$$

Case 2. $a \in B$

We write b instead of a for this case. We will extend ι to $\iota' \in \mathcal{I}$ such that ι' has b in its domain.

Using Case 1, we may assume that A' contains all elements of the form $b'b^k$ ($b' \in B'$, $k \in \mathbb{Z}$) such that $b'b^k \in A$.

Next we need several lemmas. For $S \subseteq B$, let $B'\langle S\rangle$ denote the subgroup of B generated by $B' \cup S$.

Lemma 6.9. *There exists a sequence $(b_q)_{q\in\mathbb{Q}}$ of elements of B such that $B'\langle(b_q)_{q\in\mathbb{Q}}\rangle = B'\langle b\rangle_B$.*

Proof. Let $q \in \mathbb{Q}$, and let $l \in \mathbb{Z}^\times$ and $n > 0$ be such that $q = n/l$ and $\gcd(l, n) = 1$. By axiom (14) in T_2, there are $b_q \in B$ and $\xi_q \in \Xi$ such that $\xi_q b_q^n = b^l$. We claim that the sequence $(b_q)_{q\in\mathbb{Q}}$ constructed in this way satisfies the desired properties.

Let $x \in B'\langle b\rangle_B$. Then there are $b' \in B'$, $m > 0$, and $l \in \mathbb{Z}$ such that $x^m = b'b^l$. Let $d = \gcd(m, l)$ and let $n_1 = m/d$, $n_2 = l/d$. We have $b^l = \xi^d_{n_1/n_2} b^m_{n_1/n_2}$, so $x^m = b'\xi^d_{n_1/n_2} b^m_{n_1/n_2}$. The element $b'\xi^d_{n_1/n_2}$ of B' has an mth root in B, namely $xb^{-1}_{n_1/n_2}$. By purity of B' in B, there is $\beta' \in B'$ such that $b'\xi^d_{n_1/n_2} = (\beta')^m$. Therefore,

$$x^m = (b'\xi^d_{n_1/n_2})b^m_{n_1/n_2} = (\beta')^m b^m_{n_1/n_2}$$

so we have $x = \beta' b_{n_1/n_2}$. Therefore, $x \in B'\langle(b_q)_{q\in\mathbb{Q}}\rangle$.

Conversely, let $x \in B'\langle(b_q)_{q\in\mathbb{Q}}\rangle$. Then there are $q_1, \ldots, q_n \in \mathbb{Q}$ and $k_1, \ldots, k_n \in \mathbb{Z}$ such that $x = b'b^{k_1}_{q_1} \ldots b^{k_n}_{q_n}$. For $i = 1, \ldots, n$, let $q_i = c_i/d_i$, where $c_i \in \mathbb{N}$ and $d_i \in \mathbb{Z}^\times$ are such that $\gcd(c_i, d_i) = 1$. For each q_i, there is $\xi_{q_i} \in \Xi$ such that $\xi_{q_i} b^{c_i}_{q_i} = b^{d_i}$. Thus, for each i, $b^{c_ik_i}_{q_i} = (b^{d_i}\xi^{-1}_{q_i})^{k_i}$. We have

$$x^{c_1\ldots c_n} = (b')^{c_1\ldots c_n}(\xi^{-k_1}_{q_1} \ldots \xi^{-k_n}_{q_n})b^{d_1k_1+\ldots+d_nk_n}.$$

By definition of $B'\langle b\rangle_B$, we have $x \in B'\langle b\rangle_B$. $\square$

Lemma 6.10. *Let $(b_q)_{q\in\mathbb{Q}}$ be the sequence defined in the proof of Lemma 6.9. Then $G'\langle\mu(b)\rangle_G = G'\langle(\mu(b_q))_{q\in\mathbb{Q}}\rangle$.*

Proof. Let $z \in G'\langle\mu(b)\rangle_G$. Then there are $g \in G'$, $n > 0$, and $l \in \mathbb{Z}$ such that $z^n = g\mu(b)^l$. As in the proof of Lemma 6.9, let $d = \gcd(n, l)$ and let $n_1 = n/d$, $n_2 = l/d$. We have $\mu(b)^l = \mu(\xi^d_{n_1/n_2})\mu(b_{n_1/n_2})^n$, so $z^n = g\mu(\xi^d_{n_1/n_2})\mu(b_{n_1/n_2})^n$. The element $g\mu(\xi^d_{n_1/n_2})$ of G' has an nth root in G (namely $z\mu(b_{n_1/n_2})^{-1}$), so by purity of G' in G, there is $\zeta \in G'$ such that $g\mu(\xi^d_{n_1/n_2}) = \zeta^n$. Therefore,

$$z^n = (g\mu(\xi^d_{n_1/n_2}))\mu(b_{n_1/n_2})^n = \zeta^n\mu(b_{n_1/n_2})^n.$$

Since $G_{\text{tor}} = \Gamma_{\text{tor}}$, there is $\gamma \in \Gamma$ such that $z = \gamma\zeta\mu(b_{n_1/n_2})$. Since $\Gamma \subseteq G'$, we have $z \in G'\langle(\mu(b_q))_{q\in\mathbb{Q}}\rangle$.

The proof that $G'\langle(\mu(b_q))_{q\in\mathbb{Q}}\rangle \subseteq G'\langle\mu(b)\rangle_G$ is similar to the proof that $B'\langle(b_q)_{q\in\mathbb{Q}}\rangle \subseteq B'\langle b\rangle_B$ in Lemma 6.9. $\square$

From here on, we will write $B'\langle(b_q)\rangle$ to denote the subgroup of B generated by B' and the sequence $(b_q)_{q\in\mathbb{Q}}$. Likewise, we will write $G'\langle\mu(b_q)\rangle$ instead of $G'\langle(\mu(b_q))_{q\in\mathbb{Q}}\rangle$.

Let $A^{(1)} := A'$ and for $j = 1, 2, \ldots$, let

$$A^{(j+1)} := \lambda(((K'(A^{(j)}, b, \text{Re}(\mu(b)))^{\text{rc}})^{>0}).$$

Let $A^\infty := \bigcup_{j\geq 1} A^{(j)}$. By a similar proof as in Case 2 of the proof of Theorem 5.6, we have $|A^\infty| < \kappa$.

Let $B^{(1)} := B'$, let $C^{(1)} := C'$, let $D^{(1)} := D'$, let $\mu_1 = \mu'$, and let $f_1 := \iota$. For $j > 1$, we recursively define $B^{(j)} \subseteq B$, $C^{(j)} \subseteq C$, $D^{(j)} \subseteq D$, functions μ_j, ν_j, and an ordered field isomorphism $f_j : K'(A^{(j)})^{\text{rc}} \to L'(C^{(j)})^{\text{rc}}$ with $f_j \in \mathcal{I}$. In particular, for each $j \geq 1$, we will have

$$(K'(A^{(j)})^{\text{rc}}, A^{(j)}, B^{(j)}, G', \mu_j) \in \mathcal{S}(\mathcal{M})$$

and

$$(L'(C^{(j)})^{\text{rc}}, C^{(j)}, D^{(j)}, H', \nu_j) \in \mathcal{S}(\mathcal{N}).$$

For $j \geq 1$, let $K_j := K'(A^{(j)})^{\text{rc}}$ and let $L_j := L'(C^{(j)})^{\text{rc}}$.

Suppose we have defined $B^{(1)}, \ldots, B^{(j)}$, $C^{(1)}, \ldots, C^{(j)}$, $D^{(1)}, \ldots, D^{(j)}$, $\mu_1, \ldots, \mu_j$, $\nu_1, \ldots, \nu_j$, $f_1, \ldots, f_j$. Let $d_j = \dim_{\mathbb{Q}}(v(K_j(b, \mu(b))^{\text{rc}})/v(K_j))$. We have several cases, depending on d_j. If $d_j = 0$, then by Case 1 of the Fundamental Lemma, we have $A^{(j+1)} = \lambda(K_j(b, \mu(b))^{\text{rc}}) = \lambda(K_j) = A^{(j)}$. In this case, we let $B^{(j+1)} := B^{(j)}$, $C^{(j+1)} := C^{(j)}$, $D^{(j+1)} := D^{(j)}$, $\mu_{j+1} = \mu_j$, $\nu_{j+1} := \nu_j$, and $f_{j+1} := f_j$.

If $d_j = 1$ or $d_j = 2$, then by Case 2 of the Fundamental Lemma, we have $A^{(j+1)} = A^{(j)}\langle\lambda(\tau(b,\mu(b)))\rangle_A$ for some $\mathcal{L}_{or}$-definable function τ. Let $a = \lambda(\tau(b,\mu(b)))$. Since $a \in A$ and $f_j \in \mathcal{I}$, we can apply Case 1 of this theorem to find $c \in C$ and $f_{j+1} : K_j(a)^{\mathrm{rc}} \to L_j(c)^{\mathrm{rc}}$ with $f_{j+1} \in \mathcal{I}$ extending f_j. Let $B^{(j+1)} := B^{(j)}\langle a\rangle_B$, $C^{(j+1)} := C^{(j)}\langle c\rangle_C$, $D^{(j+1)} := D^{(j)}\langle c\rangle_D$. Let μ_{j+1} be the function μ'' constructed in Case 1 and let $\nu_{j+1} = \nu''$. Note that μ_{j+1} extends μ_j and ν_{j+1} extends ν_j. The proof of Case 1 shows that $(K_{j+1}, A^{(j+1)}, B^{(j+1)}, G', \mu_{j+1}) \in \mathcal{S}(\mathcal{M})$ and $(L_{j+1}, C^{(j+1)}, D^{(j+1)}, H', \nu_{j+1}) \in \mathcal{S}(\mathcal{N})$.

This completes the inductive construction. Let $B^\infty = \bigcup_{j\geq 1} B^{(j)}$, $C^\infty = \bigcup_{j\geq 1} C^{(j)}$, $D^\infty = \bigcup_{j\geq 1} D^{(j)}$, $\mu_\infty = \bigcup_{j\geq 1}\mu_j$, and $\nu_\infty := \bigcup_{j\geq 1}\nu_j$. Let $f_\infty = \bigcup_{j\geq 1} f_j$. By construction, $B^\infty = B'\langle A^\infty\rangle_B$, and so $B^\infty \subseteq K'(A^\infty)^{\mathrm{rc}}$.

Let $K^\infty = K'(A^\infty)^{\mathrm{rc}}$. Now we show that

$$(K^\infty, A^\infty, B^\infty, G', \mu_\infty) \in \mathcal{S}(\mathcal{M}).$$

It is clear that the first 3 conditions in Definition 6.4 are satisfied by this structure. Since $(K_j, A^{(j)}, B^{(j)}, G', \mu_j) \in \mathcal{S}(\mathcal{M})$ for each j, we have $A^\infty = B^\infty \cap A$. Since $\mu_j = \mu|_{K_j}$ for each j, we have $\mu_\infty = \mu|_{K^\infty}$. Since $\mu_j(B^{(j)}) = G'$ for each $j \geq 1$, we have $\mu_\infty(B^\infty) = G'$.

By assumption, $K'(i)$ and $\mathbb{Q}(GB)$ are free over $\mathbb{Q}(G'B')$. By definition of B^∞, we have $G'B' \subseteq K^\infty(i)$. Therefore, we may apply Lemma 6.6 with $E = A^\infty$ and $X = \emptyset$ to show that $K^\infty(i)$ and $\mathbb{Q}(GB)$ are free over $\mathbb{Q}(G'B', A^\infty)$. Since $\mathbb{Q}(G'B', A^\infty) \subseteq \mathbb{Q}(G'B^\infty)$, $K^\infty(i)$ and $\mathbb{Q}(GB)$ are free over $\mathbb{Q}(G'B^\infty)$.

By a similar proof as in Case 2 of the proof of Theorem 5.6, K^∞ is closed under λ. By definition, for all $x \in B^\infty$, $f_\infty(\mu_\infty(x)) = \nu_\infty(f_\infty(x))$.

We now relabel by setting $K' = K^\infty$, $A' = A^\infty$, $B' = B^\infty$, $\mu' = \mu_\infty$, $L' = L^\infty$, $C' = C^\infty$, $D' = D^\infty$, $\nu' = \nu_\infty$, and $\iota = f_\infty$.

Next we find an element $\beta \in D$ such that β and $\nu(\beta)$ satisfy the correct equivalences modulo $D^{[m]}$ and $H^{[m]}$ respectively. We will then modify β to obtain $d \in D$ and $\iota' \in \mathcal{I}$ with $\iota'(b) = d$.

Lemma 6.11. *There is $\beta \in D$ such that for all $b' \in B'$, $k \in \mathbb{Z}$, and $m > 0$:*

1. $b^k \equiv b' \mod B^{[m]} \Leftrightarrow \beta^k \equiv \iota(b') \mod D^{[m]}$*;*

2. $(\mu(b))^k \equiv \mu(b') \mod G^{[m]} \Leftrightarrow (\nu(\beta))^k \equiv \iota(\mu(b')) \mod H^{[m]}$*.*

Proof. (1): By axiom (14) in T_2, we have $[n]B = [n]D$ for each n. Therefore, by Lemma 4.2.1 in [17], we can find an element $\beta \in D$ with the desired properties.

Let β be any element of D satisfying (1). We will show that any such β also satisfies (2) for any $b' \in B'$, $k \in \mathbb{Z}$, and $m > 0$.

(2): Fix $m > 0$. By axiom (14) in T_2, we have $\zeta_1, \ldots, \zeta_{l_m} \in \Xi$ which are the representatives of the cosets of $B^{[m]}$ in B. Since $\mathcal{N} \models T$, $\zeta_1, \ldots, \zeta_{l_m}$ are also the representatives of the cosets of $D^{[m]}$ in D. Let

$$I = \{i \in \{1, \ldots, l_m\} : \zeta_i B^{[m]} \cap A \neq \emptyset\}.$$

Since $A \subseteq B$, we have that for all $\alpha \in A$, there is $i \in \{1, \ldots, l_m\}$ such that $\alpha B^{[m]} = \zeta_i B^{[m]}$. Therefore, $I \neq \emptyset$. Now let

$$J = \{j \in \{1, \ldots, l_m\} : \zeta_j D^{[m]} \cap C \neq \emptyset\}.$$

We will show that:

(A) $I = J$;

(B) for any $x \in B$,

$$\mu(x) \in G^{[m]} \Leftrightarrow x \in \bigcup_{i \in I} \zeta_i B^{[m]};$$

(C) for any $y \in D$,

$$\nu(y) \in H^{[m]} \Leftrightarrow y \in \bigcup_{j \in J} \zeta_j D^{[m]}.$$

(2) follows from (A)-(C) as follows. Let $b' \in B'$ and $k \in \mathbb{Z}$ be such that $(\mu(b))^k \equiv \mu(b') \mod G^{[m]}$. Then $\mu(b^k(b')^{-1}) \in G^{[m]}$. By (B), $b^k(b')^{-1} \in \zeta_i B^{[m]}$ for some $i \in I$. That is, $b^k \equiv b'\zeta_i \mod B^{[m]}$. By our choice of β, we have $\beta^k \equiv \iota(b')\zeta_i \mod D^{[m]}$. By (A), $I = J$, so by (C), $\nu(\beta^k \iota(b')^{-1}) \in H^{[m]}$. Therefore, $\nu(\beta)^k \equiv \nu(\iota(b')) \mod H^{[m]}$. By our assumption that $\iota \in \mathcal{I}$, we also have $\nu(\beta)^k \equiv \iota(\mu(b')) \mod H^{[m]}$.

Assuming (A)-(C), a similar proof as in the previous paragraph shows that if $\nu(\beta)^k \equiv \iota(\mu(b')) \mod H^{[m]}$, then $(\mu(b))^k \equiv \mu(b') \mod G^{[m]}$.

Now we prove (A)-(C). To prove (A), we will show that for all $i \in \{1, \ldots, l_m\}$, $\zeta_i \in AB^{[m]}$ if and only if $\zeta_i \in CD^{[m]}$. Suppose there is $a \in A$ such that $\zeta_i \in aB^{[m]}$. By Lemma 2.10, there are $\delta \in \Delta$ and $\alpha \in A$ such that $a = \delta\alpha^m$. Therefore, $\zeta_i \delta^{-1} \in B^{[m]}$. Since Ξ is pure in B by axiom (11) in T_2, there is $\xi \in \Xi$ such that $\zeta_i \delta^{-1} = \xi^m$. Taking ι of both sides, we get $\zeta_i \delta^{-1} = \iota(\xi)^m$. We have $\delta \in C$ and $\iota(\xi) \in D$, so we also have $\zeta_i \in CD^{[m]}$. Since ι is an isomorphism, a similar proof shows that if $\zeta_i \in CD^{[m]}$, then $\zeta_i \in AB^{[m]}$.

Next we prove (B). Let $x \in B$ and suppose that $x \in \zeta_i B^{[m]}$ for some $i \in I$. Let $z \in B$ be such that $x = \zeta_i z^m$. Since $i \in I$, there are $a \in A$ and $c \in B$ such that $\zeta_i c^m = a$. Therefore, $x = a(c^{-1}z)^m$, so $\mu(x) = \mu(c^{-1}z)^m$ since $A \subseteq \ker(\mu)$ and μ is a group homomorphism on B. So $\mu(x) \in G^{[m]}$.

Conversely, suppose $\mu(x) \in G^{[m]}$. We have $z \in B$ such that $\mu(x) = \mu(z)^m = \mu(z^m)$. Since we assume that $\ker(\mu) \cap B = A$, we have $xz^{-m} \in A$. Let $i \in \{1, \ldots, l_m\}$ be such that $xB^{[m]} = \zeta_i B^{[m]}$. Then $xz^{-m} \in \zeta_i B^{[m]}$, so $\zeta_i B^{[m]} \cap A \neq \emptyset$. Therefore i is in I and $x \in \zeta_i B^{[m]}$.

The proof of (C) is similar to the proof of (B). □

We now fix an element β of D satisfying the properties in Lemma 6.11.

Lemma 6.12. *There is a sequence $(\beta_q)_{q\in\mathbb{Q}}$ of elements of D that has the following properties:*

1. *for all $q \in \mathbb{Q}$, $k \in \mathbb{Z}$, $b' \in B'$, and $m > 0$,*

$$b_q^k \equiv b' \mod B^{[m]} \Leftrightarrow \beta_q^k \equiv \iota(b') \mod D^{[m]};$$

2. *for all $q \in \mathbb{Q}$, $k \in \mathbb{Z}$, and $m > 0$,*

$$(\mu(b_q))^k \equiv \mu(b') \mod G^{[m]} \Leftrightarrow (\nu(\beta_q))^k \equiv \iota(\mu(b')) \mod H^{[m]}.$$

Proof. Let $(\xi_q)_{q\in\mathbb{Q}}$ be the sequence of elements of Ξ constructed in Lemma 6.9, and let $(b_q)_{q\in\mathbb{Q}}$ be the sequence of elements of B constructed in Lemma 6.9. Fix $q \in \mathbb{Q}$ with $q = n/l$ in lowest terms. By our choice of $b_{n/l}$, $\xi_{n/l}$, we have $b^l = \xi_{n/l} b_{n/l}^n$. Therefore, $b^l \equiv \xi_{n/l} \mod B^{[n]}$. By our choice of β, there is $\beta_{n/l} \in D$ such that $\beta^l = \beta_{n/l}^n \xi_{n/l}$. Let $\beta_q = \beta_{n/l}$. We now prove that the sequence $(\beta_q)_{q\in\mathbb{Q}}$ has the desired properties.

(1): Fix $q \in \mathbb{Q}$ with $q = n/l$ in lowest terms. Let $b' \in B'$, $k \in \mathbb{Z}$, and $m > 0$ be such that $b_q^k \equiv b' \mod B^{[m]}$. Then $b_q^{nk} \equiv (b')^n \mod B^{[nm]}$. By our choice of b_q, we have $b^l = b_{n/l}^n \xi_{n/l}$. Therefore, $b^{lk} \equiv \xi_{n/l}^k (b')^n \mod B^{[nm]}$. By our choice of β, $\beta^{lk} \equiv \xi_{n/l}^k (\iota(b'))^n \mod D^{[nm]}$. By our choice of $\beta_{n/l}$, we have

$$\beta_{n/l}^{nk} = \beta^{lk} \xi_{n/l}^{-k} \equiv \iota(b')^n \mod D^{[nm]}.$$

Since D is torsion free, $\beta_{n/l}^k \equiv \iota(b') \mod D^{[m]}$. The proof that if $\beta_q^k \equiv \iota(b') \mod D^{[m]}$, then $b_q^k \equiv b' \mod B^{[m]}$ is similar.

(2): The proof of this is similar to the proof of (2) in Lemma 6.11, replacing b by b_q and β by β_q. Note that to prove that (2) follows from (A)-(C) we need that for all $q \in \mathbb{Q}$, $k \in \mathbb{Z}$, $b' \in B'$, $m > 0$, $b_q^k \equiv b' \mod B^{[m]}$ if and only if $\beta_q^k \equiv \iota(b') \mod D^{[m]}$. But this is what we proved in (1). □

Consider the sets of formulas

$$\Sigma_1(x) := \{\iota(k_1) < x < \iota(k_2) : k_1, k_2 \in K', k_1 < b < k_2\}$$

$$\Sigma_2(x) := \{\mathcal{O}(f_2(x,\iota(k_2)), \nu(x), f_1(x,\iota(k_1))) : k_1,k_2 \in K', f_1, f_2\ \mathcal{L}_{or}\text{-definable}, \mathcal{O}(f_2(b,k_2), \mu(b), f_1(b,k_1))\}$$

$$\Sigma_3(x) := \{\iota(b')x^l \in B^{[m]} : b'b^l \in B^{[m]}, b' \in B', l \in \mathbb{Z}, m > 0\}$$

$$\Sigma_3'(x) := \{\iota(b')x^l \notin B^{[m]} : b'b^l \notin B^{[m]}, b' \in B', l \in \mathbb{Z}, m > 0\}$$

$$\Sigma_4(x) := \{\iota(\mu(b'))\nu(x)^l \in H^{[m]} : \mu(b')\mu(b)^l \in G^{[m]}, b' \in B', l \in \mathbb{Z}, m > 0\}$$

$$\Sigma_4'(x) := \{\iota(\mu(b'))\nu(x)^l \notin H^{[m]} : \mu(b')\mu(b)^l \notin G^{[m]}, b' \in B', l \in \mathbb{Z}, m > 0\}$$

Lemma 6.13. *There exists $d \in D$ satisfying $\Sigma(x)$.*

Proof. We will adjust the element $\beta \in D$ chosen above to satisfy Σ_1 and Σ_2, and show that the new element constructed in this way also satisfies Σ_3, Σ_3', Σ_4, and Σ_4'. Let

$$\Sigma(x) := \Sigma_1(x) \cup \Sigma_2(x) \cup \Sigma_3(x) \cup \Sigma_3'(x) \cup \Sigma_4(x) \cup \Sigma_4'(x).$$

Let $k_1, k_2 \in K'$ be such that $k_1 < b < k_2$. Let $y_1, y_2 \in K'$ and f_1, f_2 be $\mathcal{L}_{or}$-definable functions such that $\mathcal{M} \models \mathcal{O}(f_2(b,y_2), \mu(b), f_1(b,y_1))$.

Let $X \subseteq K^3$ be defined by

$$X := \{(x,z) \in K^{>0} \times \mathbb{S}^1(K) : \mathcal{M} \models [x \in (k_1,k_2)] \wedge \mathcal{O}(f_2(x,y_2), z, f_1(x,y_1))\}.$$

Thus, X is $\mathcal{L}_{or}(K')$-definable and $(b, \mu(b)) \in X$.

Next we prove that for all $x \in B \setminus B'$ and all polynomials p over K', we have $p(x,\mu(x)) \neq 0$ by the Mann property. Suppose for a contradiction that there is a polynomial p with coefficients in K' such that $p(x,\mu(x)) = 0$. Then $\{x, \mu(x)\}$ is a subset of GB which is algebraically dependent over $K'(i)$. By freeness of $K'(i)$ and $\mathbb{Q}(GB)$ over $\mathbb{Q}(G'B')$, $\{x,\mu(x)\}$ is algebraically dependent over $\mathbb{Q}(G'B')$. Since $(K, GB, (\gamma\xi)_{\gamma\in\Gamma,\xi\in\Xi})$ satisfies the Mann axioms for $\Gamma\Xi$, we can apply Lemma 5.12 in [13] to obtain $n, m \in \mathbb{Z}$ and $g' \in G'$, $b' \in B'$ such that $x^m\mu(x)^n = g'b'$. Since we assume that $x \in B$ and $G \cap B = \{1\}$, we have $x^m = b'$. By purity of B' in B, we have $x \in B'$, contradicting our choice of x. Therefore, $p(x,\mu(x)) \neq 0$ for any polynomial p over K'.

We claim that the set X has interior in the subspace topology on $K^{>0} \times \mathbb{S}^1(K)$. If it does not have interior, then by cell decomposition, there are $n \geq 1$ and polynomials $p_1, \ldots, p_n$ with coefficients in K' such that for every $x \in B \setminus B'$, there is i such that $p_i(x,\mu(x)) = 0$. This contradicts what we proved in the previous paragraph.

Let

$$\iota(X) := \{(x, z) \in L^{>0} \times \mathbb{S}^1(L) : \mathcal{N} \models [x \in (\iota(k_1), \iota(k_2))] \wedge \mathcal{O}(f_2(x, \iota(y_2)), z, f_1(x, \iota(y_1)))\}.$$

By similar reasoning as before, $\iota(X)$ has interior in the subspace topology on $L^{>0} \times \mathbb{S}^1(L)$. Therefore, the set

$$Y := \{(\beta^{-1}x, \nu(\beta^{-1})z) : (x, z) \in \iota(X)\}$$

also has interior as a subset of $L^{>0} \times \mathbb{S}^1(L)$. (Here the multiplication $\nu(\beta^{-1})z$ is complex multiplication.) Let $I_1 \subseteq L^{>0}$ and $I_2 \subseteq \mathbb{S}^1(L)$ be sets open in the respective subspace topologies such that $I_1 \times I_2 \subseteq Y$.

We will first show that for any $m \in \mathbb{N}$, there is $\zeta_m \in D^{[m]}$ such that $(\zeta_m, \nu(\zeta_m)) \in I_1 \times I_2$. Fix $m \in \mathbb{N}$. By axiom (7) in T_2, the graph of ν is dense in $L^{>0} \times \mathbb{S}^1(L)$. By axiom (8) in T_2, $\nu(x) = 1$ for any $x \in L \setminus D$. Therefore, $D \times \nu(D)$ is also dense in $L^{>0} \times \mathbb{S}^1(L)$. Let $S := D \times \nu(D)$. Let $\varphi_m : L \to L$ be the map $x \mapsto x^m$ and let $\psi_m : L(i) \to L(i)$ be the map $z \mapsto z^m$, where z^m denotes complex multiplication. Clearly φ_m and ψ_m are continuous, so the map $\varphi_m \times \psi_m : L \times L(i) \to L \times L(i)$ is also continuous. Since S is dense in $L^{>0} \times \mathbb{S}^1(L)$, $(\varphi_m \times \psi_m)(S)$ is also dense in $L^{>0} \times \mathbb{S}^1(L)$. In particular, there is $\zeta_m \in D^{[m]}$ such that $(\zeta_m, \nu(\zeta_m)) \in I_1 \times I_2$.

By κ-saturation of $\mathcal{N}$, there is $\zeta' \in \bigcap_{n \geq 1} D^{[n]}$ such that $(\zeta', \nu(\zeta')) \in I_1 \times I_2$. Therefore, we have $(\zeta'\beta, \nu(\zeta'\beta)) \in \iota(X)$. By definition of $\iota(X)$, it follows that

$$\iota(k_1) < \zeta'\beta < \iota(k_2) \text{ and } \mathcal{O}(f_2(\zeta'\beta, \iota(y_2)), \nu(\zeta'\beta), f_1(\zeta'\beta, \iota(y_1))).$$

Let $\eta = \zeta'\beta$. We claim that η satisfies Σ_3, Σ_3', Σ_4, and Σ_4'.

To see this, suppose $b^k \equiv b' \mod B^{[m]}$ for some $b' \in B'$, $k \in \mathbb{Z}$, and $m > 0$. By our choice of β, we have $\beta^k \equiv \iota(b') \mod D^{[m]}$. Since $\zeta' \in \bigcap_{n \geq 1} D^{[n]}$, we also have $\eta^k \equiv \iota(b') \mod D^{[m]}$. Similarly, if $b' \in B'$, $k \in \mathbb{Z}$, and $m > 0$ are such that $b^k \not\equiv b' \mod B^{[m]}$, then our choice of β gives us that $\eta^k \not\equiv \iota(b') \mod D^{[m]}$. Thus, η satisfies $\Sigma_3 \cup \Sigma_3'$.

Note that since ν is a group homomorphism on D, we have $\nu(\zeta') \in \bigcap_{n \geq 1} H^{[n]}$. Suppose $\mu(b)^k \equiv \mu(b') \mod G^{[m]}$ for some $b' \in B'$, $k \in \mathbb{Z}$, and $m > 0$. By Lemma 6.11, we have $\nu(\beta)^k \equiv \iota(\mu(b')) \mod H^{[m]}$. Since $\nu(\zeta') \in \bigcap_{n \geq 1} H^{[n]}$, we also have $\nu(\eta)^k \equiv \iota(\mu(b')) \mod H^{[m]}$. Similarly, if $b' \in B'$, $k \in \mathbb{Z}$, and $m > 0$ are such that $\mu(b)^k \not\equiv \mu(b') \mod G^{[m]}$, then Lemma 6.11 gives us that $\nu(\eta)^k \not\equiv \iota(\mu(b')) \mod H^{[m]}$. Thus, η satisfies $\Sigma_4 \cup \Sigma_4'$.

We have now proved that $\Sigma(x)$ is finitely satisfiable by an element of D. Applying κ-saturation again, we find $d \in D$ such that d satisfies $\Sigma(x)$. $\square$

Now let

$$\Phi_1((x_q)_{q\in\mathbb{Q}}) := \{\phi(\iota(k_1),\ldots,\iota(k_n),x_{q_1},\ldots,x_{q_m},\nu(x_{q_1}),\ldots,\nu(x_{q_m})) : k_1,\ldots,k_n \in K', \phi \text{ an } \mathcal{L}_{or}\text{-formula},$$
$$\mathcal{M} \models \phi(k_1,\ldots,k_n,b_{q_1},\ldots,b_{q_m},\mu(b_{q_1}),\ldots,\mu(b_{q_m})), n,m \geq 1, q_1,\ldots,q_m \in \mathbb{Q}\}$$

$$\Phi_2((x_q)_{q\in\mathbb{Q}}) := \{\iota(b')x_q^l \in B^{[m]} : b'b_q^l \in B^{[m]}, b' \in B', l \in \mathbb{Z}, m > 0, q \in \mathbb{Q}\}$$

$$\Phi_2'((x_q)_{q\in\mathbb{Q}}) := \{\iota(b')x_q^l \notin B^{[m]} : b'b_q^l \notin B^{[m]}, b' \in B', l \in \mathbb{Z}, m > 0, q \in \mathbb{Q}\}$$

$$\Phi_3((x_q)_{q\in\mathbb{Q}}) := \{\iota(\mu(b'))\nu(x_q)^l \in H^{[m]} : \mu(b')\mu(b_q)^l \in G^{[m]}, b' \in B', l \in \mathbb{Z}, m > 0, q \in \mathbb{Q}\}$$

$$\Phi_3'((x_q)_{q\in\mathbb{Q}}) := \{\iota(\mu(b'))\nu(x_q)^l \notin H^{[m]} : \mu(b')\mu(b_q)^l \notin G^{[m]}, b' \in B', l \in \mathbb{Z}, m > 0, q \in \mathbb{Q}\}$$

Let

$$\Phi((x_q)_{q\in\mathbb{Q}}) := \Phi_1((x_q)) \cup \Phi_2((x_q)) \cup \Phi_2'((x_q)) \cup \Phi_3((x_q)) \cup \Phi_3'((x_q)).$$

Lemma 6.14. *There exists a sequence $(d_q)_{q\in\mathbb{Q}}$ of elements of D that satisfies $\Phi((x_q)_{q\in\mathbb{Q}})$.*

Proof. For $m \geq 1$ and $q_1,\ldots,q_m \in \mathbb{Q}$, let $\Phi(x_{q_1},\ldots,x_{q_m})$ denote the subset of Φ consisting only of formulas whose variables are among $x_{q_1},\ldots,x_{q_m}$.

Now fix $q_1,\ldots,q_n \in \mathbb{Q}$. We want to show that there are $d_{q_1},\ldots,d_{q_n} \in D$ satisfying $\Phi(x_{q_1},\ldots,x_{q_m})$. Using κ-saturation of $\mathcal{N}$, we can then conclude that there is a sequence $(d_q)_{q\in\mathbb{Q}}$ satisfying $\Phi((x_q)_{q\in\mathbb{Q}})$.

For $1 \leq i \leq n$, let $m_i, l_i \in \mathbb{Z}$ be such that $\gcd(m_i,l_i) = 1$ and $q_i = \frac{m_i}{l_i}$. Let $N := l_1 \ldots l_n$.

By a similar proof as in Lemma 6.13, for each $m \geq 1$, there exists $d_{1/m} \in D$ such that $d_{1/m}$ satisfies $\Phi(x_{1/m})$. We can now show that $\Phi(x_{q_1},\ldots,x_{q_n})$ is satisfiable as follows. By assumption, there exists $d_{1/N} \in D$ such that $d_{1/N}$ satisfies $\Phi(x_{1/N})$. By construction of $b_{1/N}$ in Lemma 6.12, for each $i \in \{1,\ldots,n\}$, we have

$$\xi_{1/N}\xi_{q_i}^{-l_1\ldots l_{i-1}l_{i+1}\ldots l_n} b_{1/N} = b_{q_i}^{[m_i l_1\ldots l_{i-1}l_{i+1}\ldots l_n]}.$$

Since $d_{1/N}$ satisfies $\Phi_2(x_{1/N})$, we also have

$$\xi_{1/N}\xi_{q_i}^{-l_1\ldots l_{i-1}l_{i+1}\ldots l_n} d_{1/N} \in D^{[m_i l_1\ldots l_{i-1}l_{i+1}\ldots l_n]}.$$

Thus, for each $i \in \{1,\dots,n\}$, we can find $d_{q_i} \in D$ such that

$$\xi_{1/N}\xi_{q_i}^{-l_1\dots l_{i-1}l_{i+1}\dots l_n} d_{1/N} = d_{q_i}^{m_i l_1\dots l_{i-1}l_{i+1}\dots l_n}.$$

We now show that $(d_{q_1},\dots,d_{q_n})$ satisfies $\Phi(x_{q_1},\dots,x_{q_n})$.

For each $i \in \{1,\dots,n\}$, our choice of d_{q_i} gives us $\mathcal{L}_{or}(L')$-definable functions $\sigma_{1,i}$, $\sigma_{2,i}$ such that $d_{q_i} = \sigma_{1,i}(d_{1/N})$ and $\nu(d_{q_i}) = \sigma_{2,i}(\nu(d_{1/N}))$. So $(d_{q_1},\dots,d_{q_n})$ satisfies $\Phi_1(x_{q_1},\dots,x_{q_n})$. Our choice of d_{q_i} immediately gives us that d_{q_i} satisfies $\Phi_3(x_{q_i})$ and $\Phi_3'(x_{q_i})$ for each i. The proof of Lemma 6.11 shows that for each i, d_{q_i} also satisfies $\Phi_3(x_{q_i})$ and $\Phi_3'(x_{q_i})$. $\square$

Thus, we have an $\mathcal{L}_{or}$-isomorphism $\iota' : K'(b,\mu(b))^{\mathrm{rc}} \to L'(d,\mu(d))^{\mathrm{rc}}$ extending ι which takes b to d and $\mu(b)$ to $\nu(d)$. Similarly, for each $q \in \mathbb{Q}$, we have $\iota'(b_q) = d_q$ and $\iota'(\mu(b_q)) = \nu(d_q)$.

Since we chose d to satisfy Σ_3 and Σ_3', we have $\iota'(B'\langle b\rangle_B) = D'\langle d\rangle_D$. By our choice of $(b_q)_{q\in\mathbb{Q}}$, we have $B'\langle b\rangle_B = B'\langle (b_q)_{q\in\mathbb{Q}}\rangle$. Therefore, $D'\langle d\rangle_D = D'\langle (d_q)_{q\in\mathbb{Q}}\rangle$.

Next we check that $\iota'(G'\langle\mu(b)\rangle_G) = H'\langle\nu(d)\rangle_H$. Let $z \in G'\langle\mu(b)\rangle_G$. Then there are $k \in \mathbb{Z}$, $g' \in G'$, and $m > 0$ such that $z^m = g'\mu(b)^k$. Therefore, $\mu(b)^k \equiv (g')^{-1} \mod G^{[m]}$. Since ι' extends ι, we have $\iota'(B') = G'$. Thus, there is $b' \in B'$ such that $\mu(b') = g'$. Since d satisfies Σ_4, we have $\nu(d)^k \equiv \iota(g')^{-1} \mod H^{[m]}$. Let $h \in H$ be such that $\nu(d)^k\iota(g') = h^m$. Since ι' is an $\mathcal{L}_{or}$-isomorphism which takes $\mu(b)$ to $\nu(d)$, we have

$$\iota'(z)^m = \iota'(g'\mu(b)^k) = \iota'(g')(\nu(d))^k = h^m.$$

Therefore, $\iota'(z)h^{-1} \in H_{\mathrm{tor}} = \Gamma_{\mathrm{tor}}$. Since $\Gamma \subseteq H$, we have $\iota'(z) \in H$. By definition of $H'\langle\nu(d)\rangle_H$, we have $\iota'(z) \in H'\langle\nu(d)\rangle_H$. This proves that $\iota(G'\langle\mu(b)\rangle_G) \subseteq H'\langle\nu(d)\rangle_H$. The proof that $H'\langle\nu(d)\rangle_H \subseteq \iota(G'\langle\mu(b)\rangle_G)$ is similar.

Next we will show that

$$\mathcal{M}'' := (K'(b,\mathrm{Re}(\mu(b)))^{\mathrm{rc}}, A', B'\langle b\rangle_B, G'\langle\mu(b)\rangle_G, \mu|_{K'(b,\mathrm{Re}(\mu(b)))^{\mathrm{rc}}})$$

is in $\mathcal{S}(\mathcal{M})$, that

$$\mathcal{N}'' := (L'(d,\mathrm{Re}(\nu(d)))^{\mathrm{rc}}, C', D'\langle d\rangle_D, H'\langle\nu(d)\rangle_G, \nu|_{L'(d,\mathrm{Re}(\nu(d)))^{\mathrm{rc}}})$$

is in $\mathcal{S}(\mathcal{N})$, and that $\iota' : K'(b,\mathrm{Re}(\mu(b)))^{\mathrm{rc}} \to L'(d,\mathrm{Re}(\nu(d)))^{\mathrm{rc}}$ is an isomorphism of these structures. Unless otherwise noted, the proof that $\mathcal{N}''$ satisfies the conditions in Definition 6.4 is the same as the proof that $\mathcal{M}''$ satisfies these conditions. Let $K'' = K'(b,\mathrm{Re}(\mu(b)))^{\mathrm{rc}}$ and let $\mu'' = \mu|_{K''}$. Similarly, let

$L'' = L'(d, \mathrm{Re}(\nu(d)))^{\mathrm{rc}}$ and let $\nu'' = \nu|_{L''}$. It is clear that the first three conditions in Definition 6.4 hold for both $\mathcal{M}''$ and $\mathcal{N}''$.

We check that $A' = B'\langle b \rangle_B \cap A$ and $C' = D'\langle d \rangle_D \cap C$. Let $x \in B'\langle b \rangle_B \cap A$. Then there are $b' \in B'$, $k \in \mathbb{Z}$, and $n > 0$ such that $x^n = b'b^k$. Since $x \in A$, we have $b'b^k \in A$. Therefore, by the assumptions we made at the beginning of this case, we have $b'b^k \in A'$. By purity of A' in A, we have $x \in A'$. Therefore, $A' = B'\langle b \rangle_B \cap A$.

Since ι' takes A' to C', if $b' \in B'$, $k \in \mathbb{Z}$ are such that $b'b^k \in A'$, then $\iota'(b')\iota'(b)^k \in C'$. So C' contains $\bigcup_{k \in \mathbb{Z}}(D'd^k \cap C)$. Therefore, a similar proof as in the previous paragraph shows that $C' = D'\langle d \rangle_D \cap C$.

Next, we will show that $K''(i)$ and $\mathbb{Q}(GB)$ are free over $\mathbb{Q}(G'\langle \mu(b) \rangle_G B'\langle b \rangle_B)$. Let $E = \{b, \mu(b)\}$. Since $\mu(b) \in G \subseteq \mathbb{S}^1(K)$, we have $\mathrm{Re}(\mu(b)) = \frac{(\mu(b))^2+1}{2\mu(b)}$. Therefore, $\mathrm{Re}(E) \subseteq \mathbb{Q}(E)$. A similar proof as in Lemma 6.6 shows that $K''(i)$ and $\mathbb{Q}(GB)$ are free over $\mathbb{Q}(G'B', b, \mu(b))$. Thus, $K''(i)$ and $\mathbb{Q}(GB)$ are free over $\mathbb{Q}(G'\langle \mu(b) \rangle_G B'\langle b \rangle_B)$.

Next we check that $\mu''(B'\langle b \rangle_B) = G'\langle \mu(b) \rangle_G$. Let $x \in B'\langle b \rangle_B$ and let $b' \in B'$, $k \in \mathbb{Z}$, $m > 0$ be such that $x^m = b'b^k$. Since μ is a group homomorphism on B, we have $\mu(x)^m = \mu(b')\mu(b)^k$. By definition of $G'\langle \mu(b) \rangle_G$, we have $\mu(x) \in G'\langle \mu(b) \rangle_G$.

Conversely, let $z \in G'\langle \mu(b) \rangle_G$. Then there are $g' \in G'$, $k \in \mathbb{Z}$, $m > 0$ such that $z^m = g'\mu(b)^k$. Since we have $\mu(B') = G'$, let $b' \in B'$ be such that $\mu(b') = g'$. Then $z^m = \mu(b'b^k)$ since μ is a group homomorphism on B. By a similar proof as in Lemma 6.11, there is $\zeta \in \Xi$ such that $b^{-k} \equiv b'\zeta \mod B^{[m]}$. Let $x \in B$ be such that $x^m = b^k(b'\zeta)$. By definition of $B'\langle b \rangle_B$, $x \in B'\langle b \rangle_B$. Therefore, $z^m\mu(\zeta) = \mu(b'b^k\zeta) = \mu(x)^m$. By axioms (8) and (10) in T_2, $\mu(\zeta) \in G^{[m]} \cap \Gamma$. By axiom (12) in T_2, $z\mu(x)^{-1} \in \Gamma$. Let $\gamma \in \Gamma$ be such that $z\mu(x)^{-1} = \gamma$ and let $\xi \in \Xi$ be such that $\mu(\xi) = \gamma$. Then $z = \mu(x\xi)$. Since $x \in B'\langle b \rangle_B$ and $\Xi \subseteq B'$, we have $z \in \mu(B'\langle b \rangle_B)$.

By a similar proof as in Case 2 of Theorem 5.6, K'' is closed under λ by construction of $K'(A^\infty)^{\mathrm{rc}}$. Likewise, L'' is closed under λ by a similar proof as in Case 2 of Theorem 5.6.

The last thing we must check is that ι' really is an isomorphism. In particular, we must check that for all $x \in K''$, $\iota'(\mu''(x)) = \nu''(\iota'(x))$. If $x \notin B$, then $\iota'(x) \notin D$. In this case, we have $\iota'(\mu''(x)) = 1 = \nu''(\iota'(x))$. Thus, we assume that $x \in B'\langle b \rangle_B$. By construction of the sequence $(d_q)_{q \in \mathbb{Q}}$, we have $\iota'(\mu(b_q)) = \nu(d_q)$ for each $q \in \mathbb{Q}$. Moreover, the sequence $(b_q)_{q \in \mathbb{Q}}$ was constructed so that $B'\langle b \rangle_B = B'\langle (b_q)_{q \in \mathbb{Q}} \rangle$. Let $x \in B'\langle b \rangle_B$. Then there are $q_1, \ldots, q_n \in \mathbb{Q}$, $k_1, \ldots, k_n \in \mathbb{Z}$, and $b' \in B'$ such that

$$x = b' b_{q_1}^{k_1} \ldots b_{q_n}^{k_n}.$$

Since μ is a group homomorphism on B and ν is a group homomorphism on D, we have

$$\begin{aligned}\iota'(\mu''(x)) &= \iota'(\mu(b'))\iota'(\mu(b_{q_1}))^{k_1}\ldots\iota'(\mu(b_{q_n}))^{k_n}\\ &= \nu(\iota'(b'))\nu(d_{q_1})^{k_1}\ldots\nu(d_{q_n})^{k_n}\\ &= \nu(\iota'(b'b_{q_1}^{k_1}\ldots b_{q_n}^{k_n}))\\ &= \nu''(\iota'(x)).\end{aligned}$$

Therefore, we have $\iota' \in \mathcal{I}$.

Case 3. $a \in K'(\mathrm{Re}(GB))^{\mathrm{rc}}$

Since $a \in K'(\mathrm{Re}(GB))^{\mathrm{rc}}$, there are $b_1, \ldots, b_n$, $\beta_1, \ldots, \beta_n \in B$ and an $\mathcal{L}_{or}(K')$-definable function σ such that

$$a = \sigma(\mathrm{Re}(\mu(\beta_1))b_1, \ldots, \mathrm{Re}(\mu(\beta_n))b_n).$$

By using Case 2 repeatedly, we can find $\iota' \in \mathcal{I}$ extending ι such that $b_1, \ldots, b_n$, $\beta_1, \ldots, \beta_n$ are in the domain of ι'. By the construction in Case 2, $\mathrm{Re}(\mu(\beta_1)), \ldots, \mathrm{Re}(\mu(\beta_n))$ are also in the domain of ι'. Thus, ι' extends ι and has a in its domain.

Case 4. $a \in K \setminus K'(\mathrm{Re}(GB))^{\mathrm{rc}}$

As in Case 2, we first extend ι to an isomorphism

$$f_\infty : (K^\infty, A^\infty, B^\infty, G', \mu_\infty) \to (L^\infty, C^\infty, D^\infty, H', \nu_\infty)$$

such that $f_\infty \in \mathcal{I}$. By a similar proof as in Case 4 of Theorem 5.6, we can find $b \in L \setminus L'(\mathrm{Re}(HD))^{\mathrm{rc}}$ such that b realizes the same cut over L^∞ that a does over K^∞. Thus, we have an ordered field isomorphism $\iota' : K^\infty(a)^{\mathrm{rc}} \to L^\infty(b)^{\mathrm{rc}}$ extending f_∞ and taking a to b.

We will show that $(K^\infty(a)^{\mathrm{rc}}, A^\infty, B^\infty, G', \mu_\infty) \in \mathcal{S}(\mathcal{M})$. By a similar proof as in Case 4 of Theorem 5.6, $K^\infty(a)^{\mathrm{rc}}(i)$ and $\mathbb{Q}(GB)$ are free over $\mathbb{Q}(G'B^\infty)$. By definition of A^∞, $K^\infty(a)^{\mathrm{rc}}$ is closed under λ. It is easy to check that the other conditions in Definition 6.4 hold. Since $(K^\infty(a)^{\mathrm{rc}}, A^\infty, B^\infty, G', \mu_\infty) \in \mathcal{S}(\mathcal{M})$, we have $K^\infty(a)^{\mathrm{rc}} \cap B = B^\infty$. Since $f_\infty(\mu_\infty(x)) = \nu_\infty(f_\infty(x))$ for all $x \in K^\infty$, we also have $\iota'(\mu'(x)) = \nu'(\iota'(x))$ for all $x \in K^\infty(a)^{\mathrm{rc}}$.

This completes the proof that $\mathcal{I}$ is a back-and-forth system.

6.2 Predicate-near model completeness

In this section, we will prove Theorem C. As in Section 5.2.1, in order to prove this, we first introduce the notion of a special formula for our language. We define the U-restriction and V-restriction of a formula in the same way as the V-restriction is defined in Definition 5.8 (replacing V by U for the U-restriction). We define the P-restriction of a formula in the same way as in Definition 2.22.

Definition 6.15. A special formula in x with parameters from S is an $\mathcal{L}(\Gamma, \Xi)$-formula of the form

$$\exists y \exists w \exists z (U(y) \wedge V(w) \wedge P(z) \wedge \theta^1_U(y) \wedge \theta^2_V(w) \wedge \theta^3_P(z) \wedge \phi(x, y, z, w, \mu(w)))$$

where y, w are tuples of variables, $\theta^1(y)$ is an $\mathcal{L}_{om}(\Delta)$-formula, $\theta^2(w)$ is an $\mathcal{L}_{om}(\Xi)$-formula, $\theta^1_U(y)$ is the U-restriction of $\theta^1(y)$, $\theta^2_V(w)$ is the V-restriction of $\theta^2(w)$, $\theta^3_P(z)$ is the P-restriction of $\theta^3(z)$, and $\phi(x, y, w, \mu(w))$ is an $\mathcal{L}_{or}(\Gamma, \Xi, \mu, S)$-formula.

By a special formula (in x), we mean a special $\mathcal{L}(\Gamma, \Xi)$-formula in x with parameters from $\emptyset$.

For the rest of this section, whenever we refer to a special formula, we will mean an $\mathcal{L}(\Gamma, \Xi)$-formula as in the previous definition.

Now let $\mathcal{M} := (K, G, A, B, (\gamma)_{\gamma \in \Gamma}, (\delta)_{\delta \in \Delta}, (\xi)_{\xi \in \Xi}, \mu)$ be a model of T_2. Let $Y, C \subseteq K$. In analogy with Section 5.2.1, we define the *special type of* Y *over* C, denoted $\mathrm{sptp}^{\mathcal{M}}(Y|C)$, to be the set of special formulas with parameters from C satisfied by Y in $\mathcal{M}$.

Let $\mathcal{A}$, $\mathcal{B}$ be $\mathcal{L}(\Gamma, \Xi)$-structures. Let $Y, C \subseteq A$ and fix an injective function $f : C \to B$. For a sublanguage $\mathcal{L}'$ of $\mathcal{L}(\Gamma, \Xi)$, $\mathrm{tp}^{\mathcal{A}}_{\mathcal{L}'}(Y|C)$ will denote the $\mathcal{L}'(C)$-type of Y. Our definition of $f(\mathrm{tp}^{\mathcal{A}}_{\mathcal{L}'}(Y|C))$ is the same as in Section 5.2.1; that is, we set

$$f(\mathrm{tp}^{\mathcal{A}}_{\mathcal{L}'}(Y|C)) := \{\phi(x, f(c)) : \phi(x, z) \text{ an } \mathcal{L}'\text{-formula}, c \in C^{|y|}, \phi(x, c) \in \mathrm{tp}^{\mathcal{A}}_{\mathcal{L}'}(Y|C)\}.$$

In a similar fashion as in Section 5.2.1, we define $f(\mathrm{sptp}^{\mathcal{A}}(Y|C))$ by

$$f(\mathrm{sptp}^{\mathcal{A}}(Y|C)) := \{\phi(x, f(c)) : \phi(x, z) \text{ a special formula}, c \in C^{|y|}, \phi(x, c) \in \mathrm{sptp}^{\mathcal{A}}(Y|C)\}.$$

The next lemma is the analogue of Lemma 5.10 for the theory T_2. As in Section 5.2.2, this is the main lemma we will use in proving Theorem C.

Lemma 6.16. *Each $\mathcal{L}(\Gamma, \Xi)$-formula $\phi(x)$ is equivalent in T_2 to a Boolean combination of special $\mathcal{L}(\Gamma, \Xi)$-formulas in x.*

Proof. Let $\mathcal{M}, \mathcal{N} \models T$, with

$$\mathcal{M} := (K, A, B, G, \mu, (\gamma)_{\gamma\in\Gamma}, (\delta)_{\delta\in\Delta}, (\xi)_{\xi\in\Xi})$$

and

$$\mathcal{N} := (L, C, D, H, \nu, (\gamma)_{\gamma\in\Gamma}, (\delta)_{\delta\in\Delta}, (\xi)_{\xi\in\Xi}).$$

Let $\alpha \in M^m$ and $\beta \in N^m$ satisfy the same special formulas in x. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ and let $\beta = (\beta_1, \ldots, \beta_m)$.

Note that by axioms (8), (10), and (14) in T_2, for every $n > 0$, there are $k_n \geq 1$ and $\gamma_{n1}, \ldots, \gamma_{nk_n} \in \Gamma$ such that for all $g \in G$, g is equivalent to one of $\gamma_{n1}, \ldots, \gamma_{nk_n}$ modulo $G^{[n]}$. Since α and β satisfy the same special formulas, for each n and each $i, j \in \{1, \ldots, k_n\}$,

$$\mathcal{M} \models \exists y \in B(\gamma_{ni} = \gamma_{nj}\mu(y)^n) \Leftrightarrow \mathcal{N} \models \exists y \in D(\gamma_{ni} = \gamma_{nj}\mu(y)^n).$$

For $n > 0$, let ϕ_n be the sentence

$$\forall x \exists y \left(\mu(x) = \gamma_{n1}\mu(y)^n \vee \ldots \vee \mu(x) = \gamma_{nk_n}\mu(y)^n\right).$$

Since α, β satisfy the same special formulas, for each n, we have $\mathcal{M} \models \phi_n \Leftrightarrow \mathcal{N} \models \phi_n$. Therefore, we have $[n]G = [n]H$ for each $n > 0$.

Using the assumption that α, β satisfy the same special formulas, it is easy to check that for all $\gamma \in \Gamma$ and $n > 0$, γ is an nth power in G if and only if γ is an nth power in H. Likewise, for all $\xi \in \Xi$ and $n > 0$, ξ is an nth power in B if and only if ξ is an nth power in D. Therefore, we have a back-and-forth system $\mathcal{I}$ between $\mathcal{M}$ and $\mathcal{N}$ as constructed in Theorem 6.8.

We will show that $\mathrm{tp}^{\mathcal{M}}(\alpha) = \mathrm{tp}^{\mathcal{N}}(\beta)$ by finding $\iota \in \mathcal{I}$ with $\iota(\alpha) = \beta$. We may assume that $\{\alpha_1, \ldots, \alpha_r\}$ is a subset of $\{\alpha_1, \ldots, \alpha_m\}$ that is maximal with respect to being algebraically independent over $\mathbb{Q}(GB)$. Thus, there are $n > 0$ and $b \in B^n$ such that for $j \in \{r+1, \ldots, m\}$, there is an $\mathcal{L}_{or}$-definable function σ_j such that $\sigma_j(\mu(b), b, \alpha_1, \ldots, \alpha_r) = \alpha_j$. Again, since α and β satisfy the same special formulas, it can be shown that $\{\beta_1, \ldots, \beta_r\}$ is algebraically independent over $\mathbb{Q}(HD)$ by a similar argument as in Theorem 3.8 in [13].

Since $\{\alpha_1, \ldots, \alpha_r\}$ and $\{\beta_1, \ldots, \beta_r\}$ are algebraically independent over $\mathbb{Q}(GB)$ and $\mathbb{Q}(HD)$ respectively, we have $\alpha_i \notin B$ and $\beta_i \notin D$ for each $i \in \{1, \ldots, r\}$. In particular, for $i \in \{1, \ldots, r\}$, $\mu(\alpha_i) = \mu(\beta_i) = 1$.

Let $\alpha' = (\alpha_1, \ldots, \alpha_r)$ and let $\beta' = (\beta_1, \ldots, \beta_r)$. Throughout this proof, let $F = \mathbb{Q}(\mathrm{Re}(\Gamma\Xi))^{\mathrm{rc}}$.

We first construct $A^\infty \subseteq A$ in a similar way as in Case 2 of Theorem 6.8. Let $A^{(1)} = \Delta$ and let $B^{(1)} = \Xi$.

Note that $A^{(1)}$ and $B^{(1)}$ are pure subgroups of A and B respectively. Let $K_1 := F(A^{(1)})^{\mathrm{rc}}$. By a similar proof as in Lemma 2.17, $\lambda((K_1)^{>0}) = A^{(1)}$.

Now suppose that we have defined $A^{(j)}$ and $B^{(j)}$ and for $K_j := F(A^{(j)})^{\mathrm{rc}}$, $\lambda(K_j^{>0}) = A^{(j)}$. We now construct $A^{(j+1)}$ and $B^{(j+1)}$. To do this, we inductively build sequences of sets $X_1, \ldots, X_{r+n} \subseteq A$ and $Y_1, \ldots, Y_{r+n} \subseteq B$. First let $E_1 = K_j(\alpha_1)^{\mathrm{rc}}$ and let

$$X_1 := \lambda(E_1).$$

Let $d_j^1 = \dim_{\mathbb{Q}}(v(E_1)/v(K_j))$. If $d_j^1 = 0$, then by the Fundamental Lemma, we have $X_1 = A^{(j)}$. In this case, we take $Y_1 = B^{(j)}$. If $d_j^1 = 1$, then by the Fundamental Lemma, we have $X_1 = A^{(j)}\langle\lambda(\tau_1(\alpha_1))\rangle_A$ for some $\mathcal{L}_{or}(K_j)$-definable function τ_1. Let $a_1 = \lambda(\tau_1(\alpha_1))$ and let $Y_1 = B^{(j)}\langle a_1\rangle_B$.

Now let $E_2 = E_1(\alpha_2)^{\mathrm{rc}}$ and let

$$X_2 := \lambda(E_2).$$

Let $d_j^2 = \dim_{\mathbb{Q}}(v(E_2)/v(E_1))$. First suppose $d_j^2 = 1$. In this case, by the Fundamental Lemma, we have $X_2 = X_1\langle\lambda(\tau_2(\alpha_2))\rangle_A$ for some $\mathcal{L}_{or}(E_1)$-definable function τ_2. Therefore, $X_2 = A^{(j)}\langle\lambda(\tau_1(\alpha_1)), \lambda(\tau_2(\alpha_2))\rangle_A$. Using Lemma 2.15, we can find an $\mathcal{L}_{or}(E_1)$-definable function τ_3 such that $X_2 = A^{(j)}\langle\lambda(\tau_3(\alpha_1, \alpha_2))\rangle_A$. Let $a_2 = \lambda(\tau_3(\alpha_1, \alpha_2))$ and let $Y_2 = Y_1\langle a_2\rangle_B$. Note that $E_2 = K_j(\alpha_1, \alpha_2)^{\mathrm{rc}} = F(A^{(j)}, \alpha_1, \alpha_2)^{\mathrm{rc}}$.

If $d_j^2 = 0$, then by the Fundamental Lemma, $X_2 = \lambda(E_1) = X_1$. In this case, let $Y_2 = Y_1$.

Continue in this manner until we have constructed $X_1, \ldots, X_r \subseteq A$, $Y_1, \ldots, Y_r \subseteq B$, and real closed fields $E_1, \ldots, E_r \subseteq K$ such that $\lambda(E_j) = X_j$ for all $j \in \{1, \ldots, r\}$, $E_r = F(A^{(j)}, \alpha')^{\mathrm{rc}}$, and

$$X_r = A^{(j)}\langle\tau(\alpha')\rangle_A$$

for some $\mathcal{L}_{or}(E_r)$-definable function τ.

Now let $E_{r+1} = E_r(b_1, \mu(b_1))^{\mathrm{rc}}$ and let

$$X_{r+1} := \lambda(E_{r+1}).$$

Let $d_j^{r+1} = \dim_{\mathbb{Q}}(v(E_{r+1})/v(E_r))$. First suppose $d_j^{r+1} = 1$ or $d_j^{r+1} = 2$. By the Fundamental Lemma and Lemma 2.15 in [5], there is an $\mathcal{L}_{or}(E_r)$-definable function ϕ_1 such that

$$\lambda(E_{r+1}) = A^{(j)}\langle\lambda(\phi_1(\alpha', b_1, \mu(b_1)))\rangle_A.$$

Let $a_{r+1} = \lambda(\phi_1(\alpha', b_1, \mu(b_1)))$ and let $Y_{r+1} = Y_r\langle a_{r+1}\rangle_B$. If $d_j^{r+1} = 0$, then we have $X_{r+1} = X_r$. In this case, we take $Y_{r+1} = Y_r$.

Again, continue in this manner until we have constructed $X_{r+1}, \dots, X_{r+n} \subseteq A$, $Y_{r+1}, \dots, Y_{r+n} \subseteq B$, and real closed fields $E_{r+1}, \dots, E_{r+n} \subseteq K$ such that $\lambda(E_j) = X_j$ for all $j \in \{r+1, \dots, r+n\}$, $E_{r+n} = F(A^{(j)}, \alpha', b, \mu(b))^{\mathrm{rc}}$, and

$$X_{r+n} = A^{(j)}\langle \phi(\alpha', b, \mu(b))\rangle_A$$

for some $\mathcal{L}_{or}(E_{r+n})$-definable function ϕ.

Let $A^{(j+1)} = X_{r+n}$, $B^{(j+1)} = Y_{r+n}$, and $K_{j+1} = E_{r+n} = F(A^{(j+1)})$. By construction, $\lambda(K_{j+1}) = A^{(j+1)}$.

This completes the inductive construction. Now let $A^\infty = \bigcup_{j\geq 1} A^{(j)}$ and $B^\infty = \bigcup_{j\geq 1} B^{(j)}$. As in Case 2 of Theorem 6.8, we have $B^\infty = \Xi\langle A^\infty\rangle_B$ by construction. As proved in Section 6.1.2, the structure

$$(F, \Delta, \Xi, \langle\Gamma\rangle_G, \mu|_F)$$

is in $\mathcal{S}(\mathcal{M})$. Therefore, by a similar proof as in Case 2 of Theorem 6.8, the structure

$$(F(A^\infty)^{\mathrm{rc}}, A^\infty, B^\infty, \langle\Gamma\rangle_G, \mu|_{F(A^\infty)^{\mathrm{rc}}})$$

is in $\mathcal{S}(\mathcal{M})$.

Now let

$$X = \{b' b_1^{k_1} \dots b_n^{k_n} \in A : b' \in B^\infty, k_1, \dots, k_n \in \mathbb{Z}\}.$$

Let $x \in X$. Since $B^\infty = \Xi\langle A^\infty\rangle_B$, there are $\xi \in \Xi$, $a \in A^\infty$, and an $\mathcal{L}_{or}$-definable function σ such that $x = \sigma(\xi, a, b_1, \dots, b_n)$. By definition of X, we have $X \subseteq A$. Therefore, $x = \lambda(x) = \lambda(\sigma(\xi, a, b_1, \dots, b_n))$. Since $a \in A^\infty$, there is $j \geq 1$ such that $a \in A^{(j)}$. By definition of A^∞, we have $x \in A^\infty$. Therefore, $X \subseteq A^\infty$.

Let f be the function that takes α_i to β_i for $i \in \{1, \dots, m\}$.

For each $i \in \{1, \dots, n\}$, we use Lemma 6.9 to construct a sequence $(b_{i,q})_{q\in\mathbb{Q}}$ of elements of B such that $B^\infty\langle(b_{i,q})_{q\in\mathbb{Q}}\rangle = B^\infty\langle b_i\rangle_B$. By a similar proof as in Lemma 5.10, $\mathrm{sptp}^{\mathcal{M}}(\mu(b_{i,q})_{q\in\mathbb{Q},1\leq i\leq n}, (b_{i,q})_{q\in\mathbb{Q},1\leq i\leq n}, A^\infty, B^\infty|\alpha)$ is finitely satisfiable in $\mathcal{N}$ when each α_j is replaced by the corresponding β_j.

Let

$$p = \mathrm{sptp}^{\mathcal{M}}((\mu(b_{i,q})_{\substack{q\in\mathbb{Q}\\1\leq i\leq n}}, (b_{i,q})_{\substack{q\in\mathbb{Q}\\1\leq i\leq n}}, A^\infty, B^\infty|\alpha).$$

By κ-saturation of $\mathcal{N}$, there are subsets $C^\infty \subseteq C$, $D^\infty \subseteq D$, and a sequence $(d_{i,q})_{q\in\mathbb{Q},1\leq i\leq n}$ of elements of D such that

$$f(p) \subseteq \mathrm{sptp}^{\mathcal{N}}((\nu(d_{i,q})_{\substack{q\in\mathbb{Q}\\1\le i\le n}}, (d_{i,q})_{\substack{q\in\mathbb{Q}\\1\le i\le n}}, C^{\infty}, D^{\infty}|\beta). \tag{$*$}$$

In particular, letting

$$p' = \mathrm{tp}^{\mathcal{M}}_{\mathcal{L}_{or}}((\mu(b_{i,q})_{\substack{q\in\mathbb{Q}\\1\le i\le n}}, (b_{i,q})_{\substack{q\in\mathbb{Q}\\1\le i\le n}}, A^{\infty}, B^{\infty}|\alpha)),$$

we have that

$$f(p') = \mathrm{tp}^{\mathcal{N}}_{\mathcal{L}_{or}}((\nu(d_{i,q})_{\substack{q\in\mathbb{Q}\\1\le i\le n}}, (d_{i,q})_{\substack{q\in\mathbb{Q}\\1\le i\le n}}, C^{\infty}, D^{\infty}|\beta). \tag{$**$}$$

By our choice of sequence $(b_{i,q})_{q\in\mathbb{Q},1\le i\le n}$, for each $i \in \{1,\dots,n\}$ and each $q \in \mathbb{Q}$ with $q = m/l$ in lowest terms, there is $\xi_q \in \Xi$ such that $\xi_q(b_{i,q})^m = b_i^l$. Since $\Xi \subseteq B^{\infty}$, we also have $\Xi \subseteq D^{\infty}$ by our choice of D^{∞}. By $(**)$, for each $i \in \{1,\dots,n\}$, there is $d_i \in D$ such that for each $q \in \mathbb{Q}$ with $q = m/l$ in lowest terms, we have $d_i^l = \xi_{m/l}(d^i_{m/l})^m$.

Let $Y = \{d'd_1^{k_1}\dots d_n^{k_n} \in C : d' \in D^{\infty}, k_1,\dots,k_n \in \mathbb{Z}, b'b_1^{k_1}\dots b_n^{k_n} \in A\}$.

Let

$$K'' = F(\alpha', (b_{i,q})_{\substack{q\in\mathbb{Q}\\1\le i\le n}}, (\mu(b_{i,q}))_{\substack{q\in\mathbb{Q}\\1\le i\le n}}, B^{\infty})^{\mathrm{rc}}, A'' = A^{\infty}, B'' = B^{\infty}\langle(b_{i,q})_{\substack{q\in\mathbb{Q}\\1\le i\le n}}\rangle,$$

$$G'' = \Gamma\langle\mu(b_{i,q})_{\substack{q\in\mathbb{Q}\\1\le i\le n}}\rangle, \mu'' = \mu|_{K''}.$$

$$L'' = F(\beta', (d_{i,q})_{\substack{q\in\mathbb{Q}\\1\le i\le n}}, (\mu(d_{i,q}))_{\substack{q\in\mathbb{Q}\\1\le i\le n}}, D^{\infty})^{\mathrm{rc}}, C'' = C^{\infty}, D'' = D^{\infty}\langle(d_{i,q})_{\substack{q\in\mathbb{Q}\\1\le i\le n}}\rangle,$$

$$H'' = \Gamma\langle\mu(d_{i,q})_{\substack{q\in\mathbb{Q}\\1\le i\le n}}\rangle, \nu'' = \nu|_{L''}.$$

Since $X \subseteq A^{\infty}$, we have $X \subseteq K''$. By $(*)$, we also have $Y \subseteq C^{\infty} \subseteq D^{\infty}$. Therefore, we also have $Y \subseteq L''$.

By $(**)$, we have an ordered field isomorphism $\iota : K'' \to L''$ which takes $b_{i,q}$ to $d_{i,q}$ and $\mu(b_{i,q})$ to $\nu(d_{i,q})$ for each $i \in \{1,\dots,n\}$ and $q \in \mathbb{Q}$. Moreover, ι takes A'' to C'' and B'' to D''. Since G'' is generated by Γ and $(\mu(b_{i,q}))_{q\in\mathbb{Q},1\le i\le n}$ and since $\iota(\mu(b_{i,q})) = \nu(b_{i,q})$ for each $q \in \mathbb{Q}$ and $i \in \{1,\dots,n\}$, it is clear that $\iota(G'') = H''$.

By Lemma 6.9, we have $B'' = B^{\infty}\langle b_1,\dots,b_n\rangle_B$. Similarly, the proof of Lemma 6.9 shows that $D'' = D^{\infty}\langle d_1,\dots,d_n\rangle_D$. Note that $\iota(b_i) = d_i$ for each $i \in \{1,\dots,n\}$ by our choice of d_i. For the rest of this proof, let $d = (d_1,\dots,d_n)$.

Lemma 6.10 shows that $G'' = \Gamma\langle\mu(b_1),\dots,\mu(b_n)\rangle_G$. Since $d_i^l = \xi_{m/l}(d^i_{m/l})^m$ for each $i \in \{1,\dots,n\}$ and $q \in \mathbb{Q}$ with $q = m/l$ in lowest terms, the proof of Lemma 6.10 also shows that $H'' = \Gamma\langle\nu(d_1),\dots,\nu(d_n)\rangle_H$.

Next we will show that $\iota(\mu(b_i)) = \nu(d_i)$ for each $i \in \{1,\dots,n\}$. Since

$$G'' = \Gamma\langle(\mu(b_{i,q}))_{\substack{q\in\mathbb{Q}\\1\le i\le n}}\rangle = \Gamma\langle\mu(b_1),\dots,\mu(b_n)\rangle_G,$$

in particular, we have $\mu(b_i) \in \Gamma\langle (\mu(b_{i,q}))_{\substack{q\in\mathbb{Q} \\ 1\le i\le n}} \rangle$ for each $i \in \{1,\dots,n\}$. Fix $i \in \{1,\dots,n\}$. There are $\gamma \in \Gamma$, $q_1,\dots,q_m \in \mathbb{Q}$, and $i_1,\dots,i_m \in \{1,\dots,n\}$ such that

$$\mu(b_i) = \gamma\mu(b^{i_1}_{q_1})\dots\mu(b^{i_m}_{q_m}).$$

By our choice of $(\nu(d_{i,q}))_{q\in\mathbb{Q},1\le i\le n}$, we have $\iota(\mu(b_{i,q})) = \nu(d_{i,q})$ for each $q \in \mathbb{Q}$ and $i \in \{1,\dots,n\}$. Therefore,

$$\iota(\mu(b_i)) = \gamma\nu(d^{i_1}_{q_1})\dots\nu(d^{i_m}_{q_m}).$$

It remains to show that $\nu(d_i) = \gamma\nu(d^{i_1}_{q_1})\dots\nu(d^{i_m}_{q_m})$. We have

$$\mathcal{M} \models \exists w \in B(\mu(w) = \gamma\mu(b^{i_1}_{q_1})\dots\mu(b^{i_m}_{q_m}) \wedge \xi_1 b^i_1 = w).$$

This is a special formula. By $(*)$, there is $z \in D$ such that $z = \xi_1 d^i_1$ and

$$\nu(z) = \gamma\nu(d^{i_1}_{q_1})\dots\nu(d^{i_m}_{q_m}).$$

We also have $d_i = \xi_1 d^i_1$, so $\nu(d_i) = \gamma\nu(d^{i_1}_{q_1})\dots\nu(d^{i_m}_{q_m})$, as desired. Therefore, $\iota(\mu(b_i)) = \nu(d_i)$.

By $(**)$, $\sigma_j(\nu(d), d, \beta_1,\dots,\beta_r) = \beta_j$ for each $j \in \{r+1,\dots,m\}$.

Next we show that $\mathcal{M}' \in \mathcal{S}(\mathcal{M})$. We proved above that $B'' = B^\infty\langle b_1,\dots,b_n\rangle_B$ and $G'' = \Gamma\langle \mu(b_1),\dots,\mu(b_n)\rangle_G$. Therefore, B'' is a pure subgroup of B and G'' is a pure subgroup of G.

Next we check that $A'' = B'' \cap A$. Let $x \in B'' \cap A$. Then there are $b' \in B^\infty$, $k_1,\dots,k_n \in \mathbb{Z}$ and $r \in \mathbb{N}$ such that $x^r = b' b_1^{k_1}\dots b_n^{k_n}$. Since $x \in A$, we also have $b' b_1^{k_1}\dots b_n^{k_n} \in A$. Therefore, $x \in X \subseteq A^\infty$. By purity of A^∞ in A, we have $x \in A''$.

To show that $K''(i)$ and $\mathbb{Q}(GB)$ are free over $\mathbb{Q}(G''B'')$, we apply Corollary 2.8 with

$$k = F((b_{i,q})_{\substack{q\in\mathbb{Q} \\ 1\le i\le n}}, (\mu(b_{i,q}))_{\substack{q\in\mathbb{Q} \\ 1\le i\le n}}, B^\infty)^{\mathrm{rc}}$$

and $X = \{\alpha_1,\dots,\alpha_r\}$.

By a similar proof as in Case 2 of Theorem 6.8, we have $\mu(B'') = G''$.

Next we check that K'' is closed under λ. Note that

$$K'' = F(\alpha', b, \mu(b), B^\infty)^{\mathrm{rc}} = F(\alpha', b, \mu(b), A^\infty)^{\mathrm{rc}}.$$

By our choice of A^∞, $F(\alpha', b, \mu(b), A^\infty)^{\mathrm{rc}}$ is closed under λ. Thus, K'' is closed under λ.

We also show that $\mathcal{N}' \in \mathcal{S}(\mathcal{N})$. We first check that C'' is a pure subgroup of C. By construction, A'' is a pure subgroup of A. Let $c' \in C''$ and suppose there are $n \in \mathbb{N}$, $c \in C$ such that $c' = c^n$. Let $a' \in A''$ be such that $\iota(a') = c'$. We will show that there is $a \in A'$ such that $a' = a^n$. By purity of A' in A, it suffices to show that there is $y \in A$ such that $a' = y^n$. Suppose for a contradiction that no such y exists. Then the following special formula holds in $\mathcal{M}$:

$$\exists x \in A((x = a') \wedge \forall y \in A(x \neq y^n)).$$

By our choice of C^∞, the following special formula holds in $\mathcal{N}$:

$$\exists x \in C((x = c') \wedge \forall y \in C(x \neq y^n)).$$

This contradicts our assumption that $c' = c^n$. Therefore, by purity of $A^\infty \in A$, there is $a \in A^\infty$ such that $a' = a^n$. Since $a \in A^\infty$, we have $\iota(a) \in C^\infty$ and

$$c' = \iota(a') = \iota(a)^n.$$

Therefore, C'' is pure in C.

We also check that $C'' = D'' \cap C$. Let $x \in D'' \cap C$. As proved above, $D'' = D^\infty\langle d_1, \ldots, d_n\rangle_D$. Thus, there are $d' \in D^\infty$, $r \in \mathbb{N}$, and $k_1, \ldots, k_n \in \mathbb{Z}$ such that

$$x^r = d' d_1^{k_1} \ldots d_n^{k_n}.$$

By assumption, we have $x \in C$, so we also have $x^r \in C$. Therefore, $x^r \in Y$. As proved above, we have $Y \subseteq C''$, so $x^r \in C''$. By purity of C'' in C, we have $x \in C''$, as desired.

To show that $L''(i)$ and $\mathbb{Q}(HD)$ are free over $\mathbb{Q}(H''D'')$, we apply Corollary 2.8 with

$$k = F((d_{i,q})_{\substack{q\in\mathbb{Q} \\ 1\le i\le n}}, (\mu(d_{i,q}))_{\substack{q\in\mathbb{Q} \\ 1\le i\le n}}, D^\infty)^{\mathrm{rc}}$$

and $X = \{\beta_1, \ldots, \beta_r\}$.

Next we show that L'' is closed under λ. Let $w \in L''$. Then there are $k \in F$ and $d' \in D^\infty$ such that $w = \sigma(k, \beta, d, \nu(d), d')$ for some $\mathcal{L}_{or}$-definable function σ. Let $b' = \iota^{-1}(d')$. Then $\iota^{-1}(w) = \sigma(k, \alpha, b, \mu(b), b')$. Let $a = \lambda(\iota^{-1}(w))$. Since K'' is closed under λ, we have $a \in A^\infty$. By definition of λ, we have $a \le \iota^{-1}(w) < a\varepsilon$.

Since ι is an $\mathcal{L}_{or}$-isomorphism, we also have $\iota(a) \leq w < \iota(a)\varepsilon$. Again by definition of λ, we have $\iota(a) = \lambda(w)$. Therefore, $\lambda(L'') \subseteq C^\infty = L'' \cap C$. So L'' is closed under λ, as desired.

Lastly, to show that ι is an isomorphism, we check that for all $x \in B''$, $\iota(\mu(x)) = \nu(\iota(x))$. The proof of this is the same as in Case 2 of Theorem 6.8, since we take B'' to be the group generated by B^∞ and the sequences $(b_{i,q})_{q\in\mathbb{Q}}$ for $1 \leq i \leq n$. $\square$

Let $\mathcal{M} := (K, A, B, G, (\gamma)_{\gamma\in\Gamma}, (\xi)_{\xi\in\Xi})$ be a model of T_2. Note that by axiom (14) in T_2, $[n]B$ is finite for each n. Since $\mu(B) = G$ and $\mu(\Xi) = \Gamma$, $[n]G$ is also finite for each n.

By a similar proof as that of Theorem 5.7, we obtain the following theorem.

Theorem 6.17. *Let $\mathcal{M} := (K, A, B, G, \mu, (\gamma)_{\gamma\in\Gamma}, (\xi)_{\xi\in\Xi})$ be a model of T_2. Then every subset of K^m definable in $\mathcal{M}$ is a Boolean combination of subsets of K^m defined by formulas of the form*

$$\exists y \exists w \exists z (U(y) \wedge V(w) \wedge P(z) \wedge \phi(x, y, z, w, \mu(w)))$$

where ϕ is a quantifier free $\mathcal{L}_{or}(K)$-formula.

Since $(\overline{\mathbb{R}}, (ae^{i\varphi})^{\mathbb{Z}} b^{\mathbb{Z}})$ is interdefinable with $(\overline{\mathbb{R}}, b^{\mathbb{Z}}, a^{\mathbb{Z}} b^{\mathbb{Z}}, (e^{i\varphi})^{\mathbb{Z}}, \rho)$, Theorem C follows.

6.3 Definable open sets

In this section, let $\mathcal{L}^* = \mathcal{L}_{or}(U, V, P, \mu)$ and let $\mathcal{L} = \mathcal{L}_{or}(U)$. Let Δ, Ξ, Γ, and ρ be as in the previous section. In particular, we assume that the graph of ρ is dense in $\mathbb{C}$ in this section. We now consider the open core of structures of the form $(\overline{\mathbb{R}}, \Delta, \Xi, \Gamma, \rho)$.

Let $\mathcal{R}^* = (\overline{\mathbb{R}}, \Delta, \Xi, \Gamma, \rho)$ and let $\mathcal{R} = (\overline{\mathbb{R}}, \Delta)$. Let $\mathcal{M}^*$ be a κ-saturated, strongly κ-homogeneous elementary extension of $\mathcal{R}^*$ (where $\kappa = |\mathbb{R}|^+$) with $\mathcal{M}^* = (M, A, B, G, \mu)$. Let $\mathcal{M}$ be the reduct of $\mathcal{M}$ to $\mathcal{L}$. Let C be a countable subset of M.

Theorem 6.18. *Every open set definable in $(\overline{\mathbb{R}}, \Delta, \Xi, \Gamma, \rho)$ is definable in $(\overline{\mathbb{R}}, \Delta)$.*

Proof. For $n \geq 1$, let

$$D_n = \{(a_1, \ldots, a_n) \in M^n : \{a_1, \ldots, a_n\} \text{ is } \mathrm{dcl}^{\mathcal{M}}_{\mathcal{L}_{or}}\text{-independent over } G \cup B \cup C\}.$$

The proof of this theorem will be similar to the proof of Theorem 5.11. Thus, we will again use Corollary 3.1 in Boxall and Hieronymi [4]. Since the topology on M is the order topology, Assumption (I) in [4] holds. We need to check that for all $n \geq 1$,

1. D_n is dense in M^n;

2. for every $a \in D_n$ and every open set $U \subseteq M^n$, if $\mathrm{tp}^{\mathcal{M}}(a|C)$ is realized in U, then $\mathrm{tp}^{\mathcal{M}}(a|C)$ is realized in $U \cap D_n$;

3. for every $a, b \in D_n$, if b realizes $\mathrm{tp}^{\mathcal{M}}(a|C)$, then b realizes $\mathrm{tp}^{\mathcal{M}^*}(a|C)$.

Fix $n \geq 1$.

(1): Since $\Gamma\Xi$ has the Mann property, GB also has the Mann property. The proof of (1) is exactly the same as the proof that condition (1) holds in Theorem 5.11, replacing S by $G \cup B \cup C$ and GA by GB.

(2): Since D_n is dense in M^n, the proof that (2) holds is the same as the proof that condition (2) holds in Theorem 5.11.

(3): Let $a, b \in D_n$ and suppose that b satisfies $\mathrm{tp}^{\mathcal{M}}(a|C)$. We want to show that b satisfies $\mathrm{tp}^{\mathcal{M}^*}(a|C)$. Since $\mathcal{M}^* \succeq (\overline{\mathbb{R}}, \Delta, \Xi, \Gamma, \rho)$, we have $\mathcal{M}^* \models T$. Let $\mathcal{I}$ denote the back-and-forth system constructed in Theorem 6.8. We will show that there is $\iota \in \mathcal{I}$ such that ι fixes C pointwise and $\iota(a_i) = b_i$ for each i. Let $f : \{a_1, \ldots, a_n\} \to \{b_1, \ldots, b_n\}$ be the function which takes a_i to b_i for each i.

Note that the sets $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are both algebraically independent over $\mathbb{Q}(GB)$ by our assumption that $a, b \in D_n$. In particular, $\mu(a_i) = \mu(b_i) = 1$ for each i.

We first construct $A^\infty \subseteq A$ as in Lemma 6.16. Let $p = \mathrm{tp}^{\mathcal{M}^*}_{\mathcal{L}_{or}}(A^\infty|a)$. By our assumption that $\mathrm{tp}^{\mathcal{M}}(a) = \mathrm{tp}^{\mathcal{M}}(b)$, $f(p)$ is finitely satisfiable by elements of A in $\mathcal{M}^*$. By κ-saturation of $\mathcal{M}^*$, there is a subset C^∞ of elements of A such that $f(\mathrm{tp}^{\mathcal{M}^*}_{\mathcal{L}_{or}}(A^\infty|a)) = \mathrm{tp}^{\mathcal{M}^*}_{\mathcal{L}_{or}}(C^\infty|b)$.

Let $F = \mathbb{Q}(\mathrm{Re}(\Gamma\Xi))$. Let

$$K' = F(a, A^\infty)^{\mathrm{rc}}, G' = \langle \Gamma \rangle_G, A' = A^\infty, B' = \Xi\langle A^\infty \rangle_B, \mu' = \mu|_{K'}$$

and let

$$L' = F(b, C^\infty)^{\mathrm{rc}}, H' = \langle \Gamma \rangle_G, C' = C^\infty, D' = \Xi\langle C^\infty \rangle_B, \nu' = \mu|_{K'}.$$

Let $\mathcal{M}' := (K', G', A', B', \mu')$ and let $\mathcal{N}' := (L', H', C', D', \nu')$.

The fact that $\mathcal{M}', \mathcal{N}'$ are in $\mathcal{S}(\mathcal{M})$ follows as in Lemma 6.16. Note that since $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are algebraically independent over $\mathbb{Q}(GB)$, we do not need to construct the sequences of $(b_{i,q})$'s and $(d_{i,q})$'s as in Lemma 6.16. $\square$

6.4 Non-dense groups

In this section, we prove Theorem D. We now consider expansions of $\overline{\mathbb{R}}$ by groups of the form $(ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}}$, where this group is not necessarily dense in $\mathbb{C}$. In the following theorem, we use algebraic conditions on the generators of the group to classify expansions of $\overline{\mathbb{R}}$ by subgroups of this form.

Theorem 6.19. *Let $a, b \in \mathbb{R}$ with $a, b > 1$ and let $\varphi \in \mathbb{R}$. Let $\Gamma := (ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}}$.*

1. *If $\{\frac{\ln(a)}{\ln(b)}, \frac{\varphi}{2\pi}, 1\}$ is linearly independent over $\mathbb{Q}$, then $(\overline{\mathbb{R}}, \Gamma)^o =_{df} (\overline{\mathbb{R}}, a^{\mathbb{Z}})$.*
2. *If $\{\frac{\ln(a)}{\ln(b)}, \frac{\varphi}{2\pi}, 1\}$ is linearly dependent over $\mathbb{Q}$ but $\dfrac{\ln(a)}{\ln(b)} \notin \mathbb{Q}$ and $\dfrac{\varphi}{2\pi} \notin \mathbb{Q}$, then $(\overline{\mathbb{R}}, \Gamma)$ defines S_ω for some nonzero $\omega \in \mathbb{R}$.*
3. *If $\varphi \in 2\pi\mathbb{Q}$ and $\dfrac{\ln(a)}{\ln(b)} \notin \mathbb{Q}$, then $(\overline{\mathbb{R}}, \Gamma) =_{df} (\overline{\mathbb{R}}, a^{\mathbb{Z}}b^{\mathbb{Z}})$.*
4. *If $\dfrac{\ln(a)}{\ln(b)} \in \mathbb{Q}$, then $(\overline{\mathbb{R}}, \Gamma) =_{df} (\overline{\mathbb{R}}, a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}})$. If in addition we have $\varphi \in 2\pi\mathbb{Q}$, then $(\overline{\mathbb{R}}, \Gamma) =_{df} (\overline{\mathbb{R}}, a^{\mathbb{Z}})$.*

The case where $a = 1$, $b \neq 1$, and $\varphi \notin 2\pi\mathbb{Q}$ was studied in Chapter 5.

Proof. (1) follows from Theorem 6.18.

(2) is Proposition 4.13.

Next we will prove (3). Let $p, q \in \mathbb{Z}$ be such that $\varphi = 2\pi\frac{p}{q}$ and $\gcd(p, q) = 1$. (Thus, $\left|(e^{i\varphi})^{\mathbb{Z}}\right| = q$.) We have

$$\Gamma = \{a^{j+kq}e^{i\frac{2\pi pj}{q}}b^l : k, l \in \mathbb{Z}, 0 \leq j \leq q-1\}.$$

Let $S = \{a^{kq}b^l : k, l \in \mathbb{Z}\} = a^{q\mathbb{Z}}b^{\mathbb{Z}}$. Then

$$\Gamma = \bigcup_{j=0}^{q-1} a^j e^{i\frac{2\pi pj}{q}} S.$$

We have

$$S = \bigcup_{j=0}^{q-1} \{x \in a^{\mathbb{Z}}b^{\mathbb{Z}} : x \equiv b^j \mod a^{q\mathbb{Z}}b^{q\mathbb{Z}}\}$$

so S is definable in $(\overline{\mathbb{R}}, a^{\mathbb{Z}}b^{\mathbb{Z}})$. Therefore, Γ is definable in $(\overline{\mathbb{R}}, a^{\mathbb{Z}}b^{\mathbb{Z}})$.

We showed in Proposition 6.1 that $a^{\mathbb{Z}}b^{\mathbb{Z}}$ is definable in $(\overline{\mathbb{R}}, \Gamma)$.

Lastly, we will prove (4). In this case, there are $p, q \in \mathbb{Z}$ with $q > 0$ such that $\gcd(p, q) = 1$ and $b = a^{p/q}$. Thus, $b^{\mathbb{Z}} = (a^{p/q})^{\mathbb{Z}}$. Letting $P = |p|$, we have $b^{\mathbb{Z}} = (a^{p/q})^{\mathbb{Z}} = (a^{P/q})^{\mathbb{Z}}$. Let $\Gamma' := a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}}$. Let $\pi : \mathbb{C}^\times \to \mathbb{S}^1$ be projection onto the unit circle.

We first show that Γ is definable in $(\overline{\mathbb{R}}, \Gamma')$. Let $S = \{a^{Pk+l}e^{i\varphi l} : k, l \in \mathbb{Z}\}$. It is not hard to show that $S = \{a^n e^{i\varphi m} : n \equiv m \mod P\}$. We have

$$\begin{aligned}(ae^{i\varphi})^{\mathbb{Z}} b^{\mathbb{Z}} &= \{a^{Pk+l+Pi/q} e^{i\varphi l} : k, l \in \mathbb{Z}, 1 \leq i \leq q\} \\ &= \bigcup_{i=1}^{q} a^{Pi/q} S.\end{aligned}$$

Since $a^{\mathbb{Z}}$ and $(e^{i\varphi})^{\mathbb{Z}}$ are definable in $(\overline{\mathbb{R}}, \Gamma')$, the groups $a^{P\mathbb{Z}}$ and $(e^{i\varphi})^{P\mathbb{Z}}$ are also definable in $(\overline{\mathbb{R}}, \Gamma')$. Therefore, S is definable in $(\overline{\mathbb{R}}, \Gamma')$, with

$$S = \bigcup_{j=0}^{P-1} \{x \in \Gamma' : |x| \equiv a^j \mod a^{P\mathbb{Z}} \text{ and } \pi(x) \equiv e^{i\varphi j} \mod (e^{i\varphi})^{P\mathbb{Z}}\}.$$

Next we show that Γ' is definable in $(\overline{\mathbb{R}}, \Gamma)$. Let $M := \{|z| : z \in \Gamma\}$. From what we proved above, $M = \{a^{Pk+l+Pi/q} : k, l \in \mathbb{Z}, 1 \leq i \leq q\}$. Next we will show that $M = (a^{1/q})^{\mathbb{Z}}$. Given $k, l \in \mathbb{Z}$ and $i \in \{1, \dots, q\}$, let $m = qPk+ql+pi$. Then clearly $a^{Pk+l+Pi/q} = a^{m/q}$, so $M \subseteq (a^{1/q})^{\mathbb{Z}}$. On the other hand, let $m \in \mathbb{Z}$. We want to show that there are $k, l \in \mathbb{Z}$ and $i \in \{1, \dots, q\}$ such that $m = qPk+ql+Pi = P(qk+i)+ql$. Since we assume that $\gcd(P, q) = 1$, by Bezout's identity, there are $K, L \in \mathbb{Z}$ such that $PK + qL = 1$. The Euclidean algorithm gives us $K' \in \mathbb{Z}$ and $j \in \{1, \dots, q\}$ such that $qK' + j = K$. Let $i \in \{1, \dots, q\}$ be such that $mj \equiv i \mod q$ and let $n \in \mathbb{Z}$ be such that $mj - i = nq$. Let $k = mK' + n$ and let $l = mL$. We have $P(qK' + j) + qL = 1$, so we also have $P(mqK' + jm) + q(mL) = m$.

Therefore, $M = (a^{1/q})^{\mathbb{Z}}$, and so the set $(a^{1/q})^{\mathbb{Z}}$ is definable in $(\overline{\mathbb{R}}, \Gamma)$. The set $a^{\mathbb{Z}} = \{x^q : x \in M\}$ is also definable in $(\overline{\mathbb{R}}, \Gamma)$. We have $\Gamma' = a^{\mathbb{Z}} \cdot \pi(\Gamma)$, so Γ' is definable in $(\overline{\mathbb{R}}, \Gamma)$.

If we additionally have $\varphi \in 2\pi\mathbb{Q}$, then $(e^{i\varphi})^{\mathbb{Z}}$ is finite. In this case, it is clear that $(\overline{\mathbb{R}}, a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}})$ and $(\overline{\mathbb{R}}, a^{\mathbb{Z}})$ are interdefinable.

$\square$

References

[1] Charalambos D. Aliprantis and Owen Burkinshaw, *Principles of Real Analysis*, 2nd ed., Academic Press, 1990.

[2] Oleg Belegradek and Boris Zilber, *The model theory of the field of reals with a subgroup of the unit circle*, J. London Math. Soc. **78** (2008), 563-579.

[3] Nicolas Bourbaki, *Algebra II: Chapters 4-7*, Elements of Mathematics, Springer, 2003.

[4] Gareth Boxall and Philipp Hieronymi, *Expansions which introduce no new open sets*, The Journal of Symbolic Logic **77** (2012), 111-121.

[5] Erin Caulfield, *On expansions of the real field by complex subgroups*, Annals of Pure and Applied Logic **168** (2017), no. 6, 1308-1334.

[6] ———, *On expansions of the real field by complex subgroups II.*

[7] Alfred Dolich, Chris Miller, and Charles Steinhorn, *Structures having o-minimal open core*, Trans. Amer. Math. Soc. **362** (2010), no. 3, 1371–1411.

[8] Lou van den Dries, *Algebraic theories with definable Skolem functions*, J. Symbolic Logic **49** (1984), no. 2, 625–629.

[9] ———, *Lectures on the model theory of valued fields*, Model theory in algebra, analysis and arithmetic, Lecture Notes in Math., vol. 2111, Springer, Heidelberg, 2014, pp. 55–157.

[10] ———, *The field of reals with a predicate for the powers of two*, Manuscripta Mathematica **54** (1985), 187-195.

[11] ———, *T-convexity and tame extensions II*, The Journal of Symbolic Logic **62** (1997), 14-34.

[12] ———, *Tame Topology and O-minimal Structures*, London Mathematical Society Lecture Note Series, vol. 248, Cambridge University Press, 1998.

[13] Lou van den Dries and Ayhan Günaydın, *The fields of real and complex numbers with a small multiplicative group*, Proceedings of the London Mathematical Society (3) **93** (2006), 43-81.

[14] Lou van den Dries and Adam H. Lewenberg, *T-convexity and tame extensions*, J. Symbolic Logic **60** (1995), no. 1, 74–102.

[15] J.-H. Evertse, *Geometry of numbers* (2017), http://www.math.leidenuniv.nl/~evertse/dio17-2.pdf. Universiteit Leiden.

[16] J.-H. Evertse, H.P. Schlickewei, and W.M. Schmidt, *Linear equations in variables which lie in a multiplicative group*, Annals of Mathematics **155** (2002), no. 3, 807-836.

[17] Ayhan Günaydın, *Model theory of fields with multiplicative groups*, Ph.D. Thesis, University of Illinois at Urbana-Champaign, 2008.

[18] Ayhan Günaydın and Philipp Hieronymi, *The real field with the rational points of an elliptic curve*, Fundamenta Mathematicae **211** (2011), 15-40.

[19] Philipp Hieronymi, *Defining the set of integers in expansions of the real field by a closed discrete set*, Proceedings of the American Mathematical Society **138** (2010), 2163-2168.

[20] Alexander S. Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, 1995.

[21] David Marker, *Model Theory: An Introduction*, Graduate Texts in Mathematics, vol. 217, Springer-Verlag, 2002.

[22] Chris Miller and Patrick Speissegger, *Expansions of the real line by open sets: o-minimality and open cores*, Fund. Math. **162** (1999), no. 3, 193–208.

[23] Anand Pillay and Charles Steinhorn, *Definable sets in ordered structures*, Bulletin of the American Mathematical Society **11** (July 1984), no. 1, 159-162.

[24] Michel Waldschmidt, *On Ramachandra's contributions to transcendental number theory*, The Riemann zeta function and related themes: papers in honour of Professor K. Ramachandra, Ramanujan Math. Soc. Lect. Notes Ser., vol. 2, Ramanujan Math. Soc., Mysore, 2006, pp. 155–179.

[25] Carl Ludwig Siegel, *Lectures on the Geometry of Numbers*, rewritten by Komaravolu Chandrasekharan with the assistance of Rudolf Suter, Springer-Verlag Berlin Heidelberg, 1989.

[26] Michael Tychonievich, *Tameness results for expansions of the real field by groups*, Ph.D thesis, The Ohio State University, 2013.

[27] Elias Zakon, *Generalized archimedean groups*, Trans. Amer. Math. Soc. **99** (1961), 21-40.

Appendix A

Lemmas for Fig. 1.1

In this section, we prove the lemmas necessary for Fig. 1.1.

A.1 Interdefinability of expansions of $\overline{\mathbb{R}}$

Lemma A.1. *If $\varphi \in \pi\mathbb{Q}$, then $(\overline{\mathbb{R}}, (ae^{i\varphi})^{\mathbb{Z}})$ is interdefinable with $(\overline{\mathbb{R}}, a^{\mathbb{Z}})$.*

Proof. To see this, note that $a^{\mathbb{Z}} = \{|z| : z \in S\}$. Conversely, since $\varphi \in \pi\mathbb{Q}$, let $\varphi = \pi\frac{j}{k}$, where j/k is in lowest terms. Note that $(ae^{i\varphi})^{\mathbb{Z}} \cap \mathbb{S}^1$ is finite, since we assume that $\varphi \in \pi\mathbb{Q}$. Let $C = (ae^{i\varphi})^{\mathbb{Z}} \cap \mathbb{S}^1$. So $(ae^{i\varphi})^{\mathbb{Z}}$ is generated by $(a^{\mathbb{Z}})^{[k]}$ together with C. Let $f : a^{\mathbb{Z}} \to \mathbb{C}$ be defined by

$$f(x) = \begin{cases} x, & x \in (a^{\mathbb{Z}})^{[k]} \\ xe^{i\varphi}, & x \in a(a^{\mathbb{Z}})^{[k]} \\ xe^{i(2\varphi)}, & x \in a^2(a^{\mathbb{Z}})^{[k]} \\ \vdots & \\ xe^{i((k-1)\varphi)}, & x \in a^{k-1}(a^{\mathbb{Z}})^{[k]} \end{cases}$$

Then $S = f(a^{\mathbb{Z}})$. Note that f is definable because C is finite. This finishes the proof that $(\overline{\mathbb{R}}, S)$ and $(\overline{\mathbb{R}}, a^{\mathbb{Z}})$ are interdefinable. □

We also obtain the following as an easy corollary of the previous lemma.

Corollary A.2. *If $\varphi \in \pi\mathbb{Q}$ and $\psi \in \pi\mathbb{Q}$, then $(\overline{\mathbb{R}}, (e^{i\varphi})^{\mathbb{Z}}(be^{i\psi})^{\mathbb{Z}})$ is interdefinable with $(\overline{\mathbb{R}}, b^{\mathbb{Z}})$.*

Next we consider expansions of $\overline{\mathbb{R}}$ by groups of the form $(ae^{i\varphi})^{\mathbb{Z}}(be^{i\psi})^{\mathbb{Z}}$.

Lemma A.3. *Let $a \neq 1$, $b \neq 1$ be elements of $\mathbb{R}^{>0}$. If $\varphi, \psi \in \pi\mathbb{Q}$, then $(\overline{\mathbb{R}}, (ae^{i\varphi})^{\mathbb{Z}}(be^{i\psi})^{\mathbb{Z}})$ is interdefinable with $(\overline{\mathbb{R}}, a^{\mathbb{Z}}b^{\mathbb{Z}})$.*

Proof. We have $a^{\mathbb{Z}}b^{\mathbb{Z}} = \{|z| : z \in (ae^{i\varphi})^{\mathbb{Z}}(be^{i\psi})^{\mathbb{Z}}\}$.

Next we show that $(ae^{i\varphi})^{\mathbb{Z}}(be^{i\psi})^{\mathbb{Z}}$ is definable in $(\overline{\mathbb{R}}, a^{\mathbb{Z}}b^{\mathbb{Z}})$. Let $\varphi = \pi\frac{p_1}{q_1}$ and $\psi = \pi\frac{p_2}{q_2}$, where both of these fractions are in lowest terms. It is not hard to show that

$$(ae^{i\varphi})^{\mathbb{Z}}(be^{i\psi})^{\mathbb{Z}} = \bigcup_{j_1=0}^{q_1-1}\bigcup_{j_2=0}^{q_2-1}(ae^{i\varphi})^{j_1}(be^{i\psi})^{j_2}a^{q_1\mathbb{Z}}b^{q_2\mathbb{Z}}.$$

Thus, if we can show that $a^{q_1\mathbb{Z}}b^{q_2\mathbb{Z}}$ is definable in $(\overline{\mathbb{R}}, a^{\mathbb{Z}}b^{\mathbb{Z}})$, we will have proved that $(ae^{i\varphi})^{\mathbb{Z}}(be^{i\psi})^{\mathbb{Z}}$ is definable in $(\overline{\mathbb{R}}, a^{\mathbb{Z}}b^{\mathbb{Z}})$. Clearly, $a^{q_1\mathbb{Z}}b^{q_2\mathbb{Z}} = (a^{q_1\mathbb{Z}}b^{\mathbb{Z}}) \cap (a^{\mathbb{Z}}b^{q_2\mathbb{Z}})$. Therefore, to show that $a^{q_1\mathbb{Z}}b^{q_2\mathbb{Z}}$ is definable in $(\overline{\mathbb{R}}, a^{\mathbb{Z}}b^{\mathbb{Z}})$, it suffices to show that $a^{q_1\mathbb{Z}}b^{\mathbb{Z}}$ and $a^{\mathbb{Z}}b^{q_2\mathbb{Z}}$ are both definable in this structure.

To see that $a^{q_1\mathbb{Z}}b^{\mathbb{Z}}$ is definable in $(\overline{\mathbb{R}}, a^{\mathbb{Z}}b^{\mathbb{Z}})$, note that

$$a^{q_1\mathbb{Z}}b^{\mathbb{Z}} = \bigcup_{j=0}^{q_1-1}\{x \in a^{\mathbb{Z}}b^{\mathbb{Z}} : x \equiv b^j \mod a^{q_1\mathbb{Z}}b^{q_1\mathbb{Z}}\}.$$

Similarly,

$$a^{\mathbb{Z}}b^{q_2\mathbb{Z}} = \bigcup_{j=0}^{q_2-1}\{x \in a^{\mathbb{Z}}b^{\mathbb{Z}} : x \equiv a^j \mod a^{q_2\mathbb{Z}}b^{q_2\mathbb{Z}}\}.$$

$\square$